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Essays in Dynamic Experimentation

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by

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All remaining errors are my own.

Essays in Dynamic Experimentation

by

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Innovation and knowledge are critical for the development of the modern economy. I design and study dynamic models for the funding of new research under different economic conditions.

In the first essay, I develops a model for the funding of R&D initiated by an entrepreneur. In the model, the funding is undertaken by a large homogeneous pool of investors. The entrepreneur can bank present investment funds for either future experimentation or diversion. R&D activities are not observable. There are two main conclusions. First, even when entrepreneurs have full bargaining power, commitment and incentive problems imply that R&D is usually inefficiently funded. Second, stronger reporting enforcement can be welfare enhancing and improves the outcomes for the entrepreneur.

In the second essay, I study funding of the projects at the early stages of the startup development. I search for the best feasible contract that can be signed by the entrepreneur and the investor. The contract provides dynamic incentives to work on the risky project in the presence of convex effort costs, private valuations, and developed credit markets. I reveal that the best feasible contract satisfies three main properties: funding is provided independent of the project failure or success; private valuations are internalized; and the work on the project does not stop until the project succeeds.

In the third essay, I study how venture capitalists provide funds to entrepreneurs to finance risky projects that exhibit diminishing returns to scale. I show that the funding rates strictly decrease in time in the full information and the observable but unverifiable information environments. In the unobservable information environment, the funding rates eventually become strictly decreasing, but they may increase in the beginning.

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Chapter I

Crowdfunding Experimentation

1. Introduction

Giving money to somebody on the Internet for the chance of receiving something they do not yet have but promise to produce with the help of your funds sounds rather risky. And yet this is how crowdfunding works. Crowdfunding is a new form of project financing. It allows entrepreneurs to receive funds from large numbers of people on the Internet. Entrepreneurs promote their ideas, solicit funding, and communicate with the public by the means of special websites. These websites are called crowdfunding platforms. Professional and amateur investors called backers visit crowdfunding platforms in search of interesting ideas. Backers fund projects in exchange for products, gifts, equity, or just out of goodwill. Crowdfunding platforms act as financial intermediaries between backers and entrepreneurs. They transfer money from the backers to the entrepreneurs while keeping a small fee and administering basic oversight.

Crowdfunding poses several challenges from the perspectives of economic theory. First, crowdfunding environment is susceptible to *moral hazard* as investors have little control over how the funds are spent by entrepreneurs. Second, entrepreneurs may not have enough *incentives* to work on their ideas efficiently when they need to share the benefits of their projects with investors. Third, the crowdfunding environment is restrictive and limited. Working on a crowdfunding project is a *dynamic learning process* that involves risk. Writing a contract between the founder and the backers of the project is complicated as there are many important details to be described. However, crowdfunding platforms aim to be accessible to general public, who are not willing or not able to get too deep into the details about the projects. They will not have time or desire to write and sign complex contracts. Thus crowdfunding is prone to be inefficient.

The first and the second challenge can be taken up by simply using the models of static moral hazard. To rise up to the third challenge, the dynamic learning problem, using the static models would not suffice. Over the years, dynamic models of moral hazard have evolved to account for the learning component of environments with continuous time and hidden actions. Therefore, the models of dynamic moral hazard, or, as they are also called, dynamic agency models, should be the perfect fit for describing the crowdfunding environments.

The main theoretical problem, then, is to use the dynamic agency framework to discover the contractual and regulatory instruments that can alleviate the inefficiency of crowdfunding, when it exists. To solve it, I create a crowdfunding model within the dynamic agency environment. Then, I (a) characterize the efficient outcome, (b) solve the model to produce the equilibrium, and (c) describe an alternative mechanism that may shift crowdfunding market closer to efficiency. I also compare different forms of crowdfunding. In the end, I reveal that the current crowdfunding regulation in the U.S. can only aggravate the inefficiency of crowdfunding, when it is present. However, it is possible that market-based mechanisms will emerge to alleviate the inefficiency, because the founders of the crowdfunded projects will benefit from improved efficiency the most. In any case, stronger reporting enforcement in crowdfunding is needed.

1.1. Motivation

The crowdfunding industry is new and growing. With the recent emergence of the equity crowdfunding, its role in the process of innovation will only become more important. A viable theoretical model of crowdfunding is needed to predict how the market will respond to the changes in crowdfunding practices and evolution of a regulatory framework. When enough data are accumulated to perform the effective empirical study of crowdfunding, the theoretical model will provide a structural framework for empirical analysis. An immediate outcome of the theoretical modeling of crowdfunding is twofold. First, it reveals which forms of crowdfunding are inefficient. Second, it allows testing possible mechanisms that may curb the inefficiency.

Methodologically, the crowdfunding market presents a good opportunity to apply and expand the framework of dynamic agency. In particular, the models of dynamic experimentation, a subset of dynamic agency models which deal with the process of financing of innovation, look very attractive to use in the modeling of crowdfunding. There are several reasons for that. The interaction between entrepreneurs and investors in crowdfunding has a clear pattern. Thus modeling it from the perspectives of game theory is straightforward. Also, crowdfunded projects are risky and require time, money, and effort to complete. Such projects have been described in dynamic experimentation literature extensively. Therefore, crowdfunding can be modeled well within the dynamic experimentation paradigm. I expand this framework by assuming the property of diminishing returns to experimentation effort and the possibility of intertemporal reallocation of funds.

This work is the first to model crowdfunding as a dynamic continuous process. Crowdfunding is usually described as a static stage game. By applying the dynamic experimentation framework to crowdfunding, I reveal the need to model the dynamic learning process about the project undertaken by the entrepreneur. Static models of crowdfunding capture only the moral hazard aspect of the inefficiency. Consequently, they tend to recommend static solutions. If it were a case, then financial hostages (see Williamson, 1983, for the explanation of how financial hostages fascilitate commitment), for example, could be used to eliminate inefficiency. However, in the dynamic setting, financial hostages would not work as intended, because incentives evolve from time period to time period based on learning. Dynamic experimentation models expose the complexities of moral hazard, adverse selection, Bayesian learning, and incentives in the process of innovation. Therefore, to create a viable model of crowdfunding, dynamics must be taken into account.

1.2. Main Results

There are three main results:

1. The first finding of this chapter, which has pragmatic and policy implications, is that audit of crowdfunding expenditures can be both socially desirable and beneficial to the entrepreneurs. Audited reporting works as a commitment device for the entrepreneur. The entrepreneur knows she will have to reveal her expenditures accumulated over a report-

ing period. She can pre-commit to reach a certain level of aggregate expenditures at the time of the reporting or suffer punishment if she does not. This way, it becomes possible to alleviate the moral hazard by having a target expense rate for every reporting period and demonstrating that these targets are being reached. The entrepreneur can set these targets such that it improves the experimentation rates for her benefit. I show that the entrepreneur captures the social surplus, therefore, what is beneficial to the entrepreneur is socially beneficial as well.

The implication is that we should expect the emergence of the audited expenditure reporting practice in crowdfunding in the near future. There are two ways it could happen: either the regulators would require it for all crowdfunded startups, or the market for the audit services targeted at the needs of crowdfunding develops. Right now, the second option looks more viable, because the regulation is heading in the opposite direction.

The introduction of the JOBS act—the main piece of crowdfunding regulation—relaxed the reporting requirements. The act was supported by the entrepreneurs and startup owners and criticized by the investors. The best explanation for this is that the costs of reporting in crowdfunding currently outweigh the benefits of transparency. If this is the case then requiring the mandatory audited reporting of expenses in crowdfunding may be damaging to the industry.

Recently, audit firms started offering services tailored to crowdfunded startups' needs. This is a positive trend suggesting that audited reporting of expenditures might eventually become a standard in crowdfunding. Given that crowdfunded startups are small, the complexity of the audit should not be an issue. The services will be limited in scope to expenditures related to particular projects, so the costs of the audit should be relatively low. Therefore, from the current perspectives, it is likely that the market solution will prevail over the regulation.

Obviously, the audited reporting of expenditures alone will not be sufficient to maximize welfare in crowdfunding. In fact, reporting will foster the culture of underexperimenting at the beginning of the reporting cycle and overexperimenting closer to the end of it. However, if the costs of reporting are relatively low then reporting is welfare enhancing.

2. The second finding is that most forms of crowdfunding are inefficient. The inefficiency stems from the unobservability of the entrepreneur's actions, the riskiness of the project, and the need to share the future surplus of the project with the investors. The riskiness and unobservability of the actions make it impossible to solve the inefficiency problem by writing incentive contracts. The need to share the surplus affects the entrepreneur's ex post incentives to experiment. Having promised to give away part of the project surplus, the entrepreneur does not feel motivated enough to efficiently conduct the experiments. There are projects that should be funded because they have the potential to generate surplus, but they will never be funded because they will be considered too risky by the investors and the entrepreneur.

   Some forms of crowdfunding, however, are efficient. In particular, if the project involves unconditional payments, then it can be efficient in theory. Unconditional payments imply that the project founder receives the funds upfront, the amount of funds is sufficient to carry on with experiments even if the project keeps failing, and that the backers only receive unconditional rewards just for participating. This way, pure donation-based crowdfunding and crowdfunding models that promise participation rewards independent of the project success can be deemed efficient. Obviously, in real world, misallocation of funds can still happen, and the possibility of fraud cannot be excluded. However, given that fraud is rare in crowdfunding, the donation-based and participation-based reward models in crowdfunding can be classified as being as close to efficiency as it is possible.

3. The third contribution is in enhancing the models of dynamic experimentation. I contribute to the dynamic experimentation literature by offering the model with convex effort costs and an intertemporal budget constraint. Convex effort costs allow me to concentrate on the interior solutions and present the effort as having the property of diminishing returns to scale. This way, I avoid the complexity of classifying the conditions associated with the

binding or relaxed incentive constraints typical for the traditional dynamic experimentation models. Intertemporal reallocation of capital is needed to model the crowdfunding market where investments are usually received upfront and in full. As a modeling tool, the intertemporal budget constraint permits the entrepreneur to be more flexible than with the interim budget constraints of traditional models.

Overall, convex costs and the intertemporal budget constraint are needed to model the crowdfunding market. Without the first property, the audited reporting of expenditures would lead to full efficiency, which is not the case in my model. Also, the equilibrium experimentation path will be trivial, coinciding with the efficient path until the project stops before it should in the efficient case. I show that this is not the case and that the equilibrium experimentation path is everywhere inefficient. Without the second property, I would not be able to model the process of receiving the funds upfront, which is essential in crowdfunding. The model I introduce is non-trivial, but analytically tractable to the point of characterizing the main properties of equilibrium time paths of all the essential variables.

1.3. Additional Results

Additionally, I characterize the properties of the efficient and the equilibrium outcomes in crowdfunding. First, the entrepreneur's budget constraint binds, which means that the entrepreneur asks for just enough funds to carry on with experiments even if the project keeps failing, but no more than that. Second, the experimentation rates in the efficient case and in the equilibrium decrease over time until they become negligible. However, the project never gets abandoned completely. Finally, there experimentation rates tend to increase the more the project promises to deliver, decrease as the monetary costs of experimentation rise, and increase when the discount rates go up.

I show that the entrepreneur's budget constraint binds, meaning that the entrepreneur will ask for enough funds to continue experimenting even if the project yields no success. The entrepreneur can save money in this money, so the funds that she receives from the backers at the beginning will be spread throughout all the experimentation sessions she will need to conduct in the future if the project keeps failing. In the event of the project success, the entrepreneur will keep the remaining funds for her personal consumption. The sum the entrepreneur receives from the backers is more than she actually expects to spend on the project, because the project might succeed at some point in time. Therefore, in equilibrium, it is expected that some of the funds will be spent on the entrepreneur's personal consumption, which is a waste from the perspectives of efficiency.

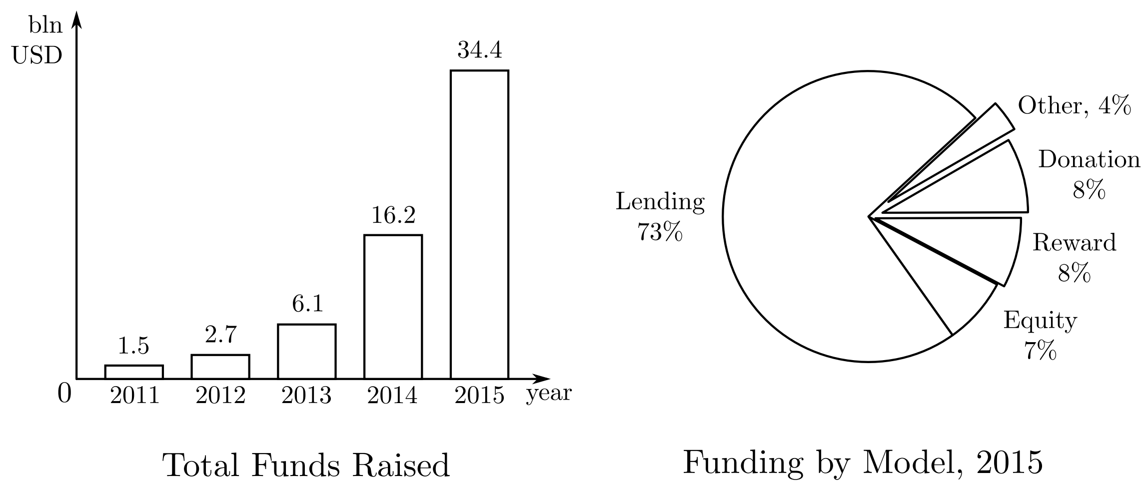
I reveal that in the efficient case and in equilibrium, the entrepreneur experiments with decreasing rates over time if she keeps seeing no success. Experimentation never really stops if the project keeps yielding no success, but eventually decreases to being negligible. In equilibrium, the entrepreneur begins with lower experimentation rates than she should to achieve efficiency, therefore, learning happens slower. It means that the posterior belief that the project is good decreases at a slower rate in equilibrium than it does in the efficient case. The parties stay optimistic about the project longer than they should.

I show that experimentation rates improve in the surplus size of the project and in the level of impatience, and decrease in marginal monetary costs of experiments. It means that, as it is expected, the projects with higher expected returns or lower costs will be worked on with greater enthusiasm. The role of impatience, represented by the discount coefficient is also clear. Relatively more impatient players will not be willing to wait long until the event of success, their current experimentation rates will be higher. These are the main comparative statics of the crowdfunding model.

1.4. Overview of the Market

In the following sections, up to the literature review, I describe the crowdfunding process, forms of crowdfunding, crowdfunding regulation, and fraud in crowdfunding. Crowdfunding is an alternative to traditional practices of project financing. According to World Bank, "crowdfunding is an Internet-enabled way for businesses or other organizations to raise money in the form of either donations or investments from multiple individuals." (InfoDev, 2013) It emerged during the financial crisis of 2007–2008 as banks, venture capitalists, and angel investors, faced with uncertainty and liquidity constraints, were less willing to fund projects and ventures. Crowdfunding

Figure I.1. Global Crowdfunding Market Size and Structure (Data from Massolution, 2015)



revolutionized capital markets by allowing general public to directly participate in risky projects and benefit from their success together with entrepreneurs.

At first, crowdfunding was a simple extension of funding by friends, family, and community members into the sphere of the Internet. Over the years, crowdfunding practices expanded, allowing startups to be funded by individuals from all over the world. According to Massolution (2015), global crowdfunding market, as measured by the total amount of raised funds, has grown from $\$1.5$ billion in 2011 to more than $\$34.4$ billion in 2015 (see Figure I.1). In comparison, the total amount of venture capital raised globally in 2011 was $\$51.7$ billion, and it reached $\$148$ billion by 2015 (EY, 2015). So in 2011, crowdfunding market was smaller than 3% as compared to the venture capital market, but by 2015, it became as big as 23% as compared to it. Unsurprisingly, North America amounts for about a half of the global crowdfunding market.

Crowdfunding is a multi-billion-dollar industry that is only expected to grow bigger. It already changes the investment environment and affects how ideas receive public attention and funding. Understanding crowdfunding and its problems from the perspectives of economic theory is important, as it will shape our view of the policies and best practices in the R&D industry.

1.5. Forms of Crowdfunding

There are several forms of crowdfunding, which can be broadly classified into two groups: donation-based models and investment-based models.

Donation-based crowdfunding : includes models of pure donations—with little to no value to the donors,—and reward-promising donations, which guarantee some specific benefits to the donors, from stickers and t-shirts to the products that will be developed by the founders.

Investment-based crowdfunding : includes equity-based models, in which backers receive equity instruments in exchange for their contributions; peer-to-peer lending, when debt instruments are sold to many private lenders online; and other models, which may constitute the exchange of derivatives, portfolios of instruments, or some other forms of investor compensation.

As it is clear from Figure I.1, lending-based instruments are responsible for almost three-quarters of the global crowdfunding market in 2015. Donation and reward-based models together account for about 16% of the market, and equity-based models comprise about 7% of it. Despite that the numbers for the equity-based form of crowdfunding are relatively modest, with the introduction of the JOBS act, equity-based crowdfunding model in the U.S. is expected to take off. It is a viable and fledgling alternative to current investment practices for many unaccredited and accredited investors[1] that simplifies the process of buying equity by getting entrepreneurs and investors closer and removing financial consultants, mutual fund managers, and personal investment managers out of the way.

1.6. The JOBS Act

Jumpstart Our Business Startups (SEC, 2012), or the JOBS act is a major attempt to regulate crowdfunding industry and provide a legal framework for the equity-based crowdfunding. Before the JOBS act, to raise capital, a U.S. startup would have had to either ask a bank for a loan, solicit donations from friends and family, or sell securities. Selling securities, in general, requires the issuer to register the securities offering with the Securities and Exchange commission (SEC) and

[1] To become an accredited investor, an individual must have a net worth excluding the primary residence higher than $1 million, or they must have a stable income of more than $200,000 a year ($300,000 for married couples).

to become a public company. There exist a number of exemptions, which allow a startup to forgo the registration and remain private.

To be exempt from SEC registration, a startup must satisfy certain requirements, but usually, a small company that intends to raise capital from a small number of *accredited* investors qualifies for one exemption or another. Companies that qualify for the exempt status are not required to publish audited annual reports as opposed to public companies. They just need to submit some general information about their securities offerings to SEC. The JOBS act eased the requirements to be exempt from SEC registration and provided legal grounds for soliciting capital online from both accredited and unaccredited investors.

Title I of the JOBS act reduced the disclosure requirements for emerging growth companies (companies with annual revenues lower than one billion dollars) and allowed them to advertise before potential investors prior to registering with SEC. This piece of regulation applies to all emerging growth companies, not only the crowdfunded ventures, providing them with relaxed conditions to raise capital.

Title II of the act allowed startups to be exempt from registration while raising unlimited amount of money from the *accredited* investors online and also advertising the investment offerings publicly. This section is important as it allowed companies to legally solicit capital online without registering offerings with SEC and only informing the Commission after the fact within two weeks of receiving the first transfer of funds.

Title III is the most important part of the JOBS act as it allowed startups to raise up to a million dollars annually from *unaccredited* investors online via crowdfunding portals registered with SEC. The disclosure rules imposed by Title III are mild (for example, principals' tax returns are not disclosed in full, only annual company financial reports must be reported, and neither an audit nor a review of the financial statements are required), but the information disclosed by startups must be available to all investors no matter how small. There are also caps on how much an unaccredited investor can spend buying shares in a startup: broadly, from 5% to 10% of annual income depending on income and net worth. This piece of legislation made equity crowdfunding model legally possible, resulting in a mini-revolution in the investment world.

Titles IV, V, VI, and VII continued the logic of deregulation and eased the requirements for low-scale IPOs, allowed access of unaccredited investors to such IPOs, raised the number of shareholders a private firm or a bank may have before they are required to register as public, and so on.

As opposed to small crowdfunded ventures, large firms acquiring capital through traditional IPOs are still subject to the same strict and precise rules regarding disclosure and capital acquisition as before the JOBS act. They still need to publicly disclose quarterly reports, they are still required to have audited and independently reviewed annual reports made available for everyone. In practice, it means that large, knowledgeable professional and institutional investors are guaranteed to have access to more information than small inexperienced unaccredited investors trying to gain by crowdfunding an emerging startup. This disproportion of power and this discontinuity of information is problematic, as people who need to be protected the most from the risks associated with the investment process are left with the least protection after the JOBS act.

Despite the claims that entrepreneurs and startup founders are most interested in the successes of their projects, without proper control from the investors, the temptation to misallocate the capital after actually receiving it may be hard for the entrepreneurs to resist.

1.7. Crowdfunding Market Structure

Crowdfunding markets, being it donation-based or equity-based crowdfunding model, consist of three major elements. First, there are startups that seek capital. Second, there are potential investors, or backers, who are interested in funding certain startups. Third, there are online portals, or platforms, that act as intermediaries between the startups and the investors by providing the required infrastructure for advertising the offerings, sending and receiving the funds, posting the reports, and for other related activities.

Under the JOBS act, portals do not share responsibility for the fraudulent behavior of the startups, however, they are required to do their best to weed out fraud. Unlike software platforms like Apple AppStore or Google Play, crowdfunding portals have little to no influence over the pricing decisions of the startups. In my model, I abstract from their existence and concentrate on the interactions between the investors and the startups founders.

The competition between crowdfunding portals, however, is potentially an exciting topic. Viotto (2015) is the first to attempt to look at crowdfunding platforms in the light of platform competition theory (see Rochet and Tirole, 2003). It is apparent today that crowdfunding portals tend to specialize and occupy certain niches of the market. For example, one of the biggest crowdfunding portals, Kickstarter, tends to post offers only from projects that have a prototype or a working concept, while their biggest competitor Indiegogo accepts riskier projects (see Indiegogo, 2014, for one of the many examples of risky projects). This product differentiation by crowdfunding portals is a promising theoretical and empirical research topic as well.

Crowdfunding process in simple. A startup founder, or the entrepreneur who seeks to obtain funds from the public designs an offering web page that will advertise the project the founder undertakes. The page is posted online on a crowdfunding portal. As a part of the offering, the entrepreneur reports the total sum of the funds she expects to raise and explains the benefits of contributing to the project for a typical investor based on the amount contributed. Usually, the offer constitutes a menu, an example of which for some imaginary project is presented in Table I.1. Potentially, the process of creating a menu of crowdfunding offers is an interesting research topic, and indeed, there are papers that seek to describe this process. For example, Hu et al. (2015) analyze product differentiation and pricing decisions in the crowdfunding market with heterogeneous agents. They describe how the optimal menu of contracts must be constructed in such an environment.

Table I.1. Example of a Benefits Menu Offered to the Project Backers

Donation	Availability	Reward
\$10	Not Limited	Thank-You Email
\$20	100	A Postcard from Our HQ
\$150	50	Early Access to the Product
\$1,000	10	Exclusive Add-Ons to the Product
\$2,000	1	Early Prototype of the Product Signed by the Founder

After the offer page is completed, the funding stage starts, and the startup begins collecting the money from the investors. Online investors who participate in crowdfunding activities invest relatively small amounts of money, sometimes less than ten dollars and rarely more than several thousand dollars. If the founder achieves their funding goal, the project is considered to be "funded", and so the founder can begin spending the acquired funds on the project to make it a success. In general, the portal keeps accepting additional funds from investors after this phase in exchange for more benefits or in donation form only. If, on the contrary, the project fails to reach the funding goal, then the money raised is returned to the investors, and the startup needs to seek alternative sources of capital or quit.

Due to lax regulatory framework, neither investors, nor online portals know what the startup company actually does with the money after the project is funded because crowdfunding is built on trust and the assumption that the entrepreneurs know better what is best for their projects and are motivated enough to achieve success. This not only creates opportunities for all sorts of scam and fraudulent behavior, but also decreases the incentives for the honest startups to do their best because the temptation to divert the funds directly influences their financing decisions.

1.8. Fraud in Crowdfunding

In 2015, in its first case in history involving crowdfunded projects, the Federal Trade Commission (FTC) took a legal action against a project creator who sought capital to finance the production of a board game, but instead diverted some of the funds and used them on himself (see FTC, 2015). It is tempting to think that the project was fraudulent from the beginning, but it may also be the case that the entrepreneur actually wanted to deliver a good product ex ante, but the availability of freely accessible funds, the lack of control, and the absence of credible commitment were the factors that made him divert the money ex post.

A different case, in which another federal commission, SEC, was involved, happened in 2015 in Texas, when company Ascenergy raised $5 mln from different crowdfunding platforms to drill oil wells on various leases in Texas, which the company claimed it had evaluated and secured. However, it turned out that the leases had neither been evaluated nor secured, and that the owners had attempted to divert the funds to their personal needs (see SEC, 2015).

There are many minor similar cases of potentially fraudulent campaigns which failed to keep the promises to the investors. In many cases, the problem is the lack of proper investigation and disclosure: in the Ascenergy case, for example, no law firm would have accepted calling the project "low risk", while that is exactly what the company had called it on the crowdfunding portals' pages. If the companies were required to disclose more information or if the investors were willing or able to learn more about the projects they wished to back, then the number of such cases would have been significantly lower.

In general, however, despite the claims of crowdfunding projects being prone to fraud, World Bank admits that "crowdfunding markets have been operating in many countries for several years with few reported instances of fraud."[2] Unfortunately, there exist no well known empirical studies of fraud in crowdfunding so far, and performing a search on the Internet reveals that the number of projects people believe to be potential crowdfunding frauds is pretty low (Cornell and Luzar, 2014). However, any empirical analysis of fraud in the crowdfunding market is complicated by the fact that crowdfunding, as any other form of investment, involves significant risk, and so when a crowdfunded project fails, it is hard to determine if this outcome was due to the fact that the funds were misallocated and so the project was not properly worked on, or because all the honest attempts by the entrepreneur to make the project work did not succeed. This unobservability of the entrepreneur's behavior is at heart of the model in this chapter.

It may not be the fraud per se, which is responsible for the problems of the crowdfunding markets, but the temptation to divert the funds and the need to provide some compensation to the backers, which create tensions between the investors and the entrepreneurs who wish to crowdfund the project. Relaxing the disclosure requirements and easing the process of soliciting funds, introduced by the JOBS act, only jeopardized the possibility to crowdfund projects efficiently.

1.9. Theoretical Literature

The model I develop in this chapter belongs to the class of models of dynamic experimentation, which is a subclass of bandit models. Dynamic experimentation models present the process of innovation as a sequence of costly experiments that may or may not succeed in the future. These

[2] InfoDev (2013), p. 10.

models were developed to explain the dynamic nature of the agency problem in the sphere of entrepreneurship and to demonstrate that static models of moral hazard alone are not sufficient to capture the important features of the problem of innovation and learning. Dynamic experimentation models became popular at the beginning of the XXI century and have a small, but prominent group of researchers working on them.

This chapter describes an extension of the model by Bergemann and Hege (2005) with some additional features. First, instead of linear effort costs, I use convex effort cost. Second, I use an intertemporal budget constraint in the model as opposed to the static budget constraint. Third, I solve the model in continuous time, not in discrete time. Finally, in my model, the negotiations between parties happen only at the beginning, funds are received upfront, and the shares are not subject to change over time as opposed to renegotiable shares and funding every time period.

It is widely accepted that the first paper that started the trend of using bandit models to explain different economic environments is Rothschild (1974). He constructed a model of a firm facing an uncertain demand and experimenting with different prices to learn about the demand properties. The model became popular and soon the whole body of literature emerged based on it. Bergemann and Välimäki (2008) provide an overview of the important models developed within this framework.

Entrepreneurship process was first modeled within the bandit framework by Bergemann and Hege (1998) and then further developed in Bergemann and Hege (2005). They assumed that there is an entrepreneur who has a project which may or may not succeed in the future. The entrepreneur does not have capital to finance the actual research on the project. Every period, the entrepreneur asks an investor for the funds in exchange for a share of the possible surplus, and the investor provides funds to the entrepreneur to conduct experiments. Experimenting is needed to find out if the project is viable, thus dynamics are essential to the learning process. In their paper, probability that the project succeeds linearly depends on the amount of money spent on experiments. The main problem they specifically addressed is that since the investor does not see what the entrepreneur is doing with the money, the entrepreneur may decide to divert the funds and pretend to conduct experiments while actually shirking. The authors showed that under certain parameter values the model may predict that the funds will be provided at a constant rate, at a decreased rate ("frontloading"), or at an increased rate ("backloading"), but in any case, the funding schedule will be suboptimal.

Hörner and Samuelson (2013) confirmed the findings of Bergemann and Hege (2005) by analyzing essentially the same model in continuous time, but with the full bargaining power given to the investor. Their approach to continuous time is different from the one employed in this chapter, because they consider that the agent can choose the length of the delay between decision-making points, while I allow the agent to control the experimentation path at every time period without delays. For the discussion of different approaches to modeling continuous time in game theory, see Simon and Stinchcombe (1989). Hörner and Samuelson (2013) assumed that the probability of success of each experiment is given, and so the cost of each experimentation session is fixed. Their model revealed that the funding rate may increase or decrease in time under Markovian decision-making assumption, and this fact prompted them to consider alternative equilibria concepts.

Another paper that studies strategic experimentation, albeit from slightly different perspectives, is Keller et al. (2005). It focuses on cooperation and competition between many identical experimenters, not on contracting between the investors and entrepreneurs. Instead of one experimenter, there are multiple agents, with replica two-armed bandits. One arm is the "safe" arm that produces a predictable payoff, the other arm is "risky": it may or may not produce a surplus if worked on. The authors characterize equilibria strategies for cooperative and strategic cases and show that free riding plays a crucial role in curtailing efficiency of the strategic experimentation with many experimenters.

There are not many papers that employ convex effort costs in dynamic agency models. Mason and Välimäki (2015) analyze the behavior of the dynamic agency model in the presence of convex costs under the assumption that parties have dissimilar discount rates, but they only consider projects that are certain to succeed. Another model that also analyzes the dynamic moral hazard with convex effort costs is by Bhaskar (2013). He studies optimal contracting schemes between a risk-neutral principal and a risk-averse agent in the presence of public signals and unobserved actions in two time periods. His paper also contains great discussion about the problems that arise in the environments in which players make both continuous and discrete choices and what role indifference has to play in such problems.

There is a paper with dynamic moral hazard and convex costs properties that belongs to a slightly different branch of literature. This branch addresses career concerns of workers who might influence their future benefits by taking unobserved actions today. Hörner and Lambert (2016) study the model, in which an agent of unknown type takes unobserved actions in continuous time to produce a stream of output. The output depends on the agent's type and actions, but it also has a random component to it. The output is consumed by the market who compensates the agent for it. The market receives its information from an intermediary via some rating system. The authors demonstrate that in the dynamic career concerns model with convex effort costs and ratings supplied by an intermediary, "excessive information depresses career concerns and distorts the agent's choices"[3] ultimately making the full disclosure environment suboptimal. They discover the optimal rating scheme that produces better results than full disclosure and describe equilibria with passive and active intermediary.

In this chapter, I build upon the main results of the previous papers on dynamic experimentation and provide additional results applicable to crowdfunding in the environment with convex experimentation costs and an intertemporal budget constraint.

1.10. Crowdfunding Literature

The economics of crowdfunding is still a fledgling topic since crowdfunding markets are still new and rapidly developing. Agrawal et al. (2013) describe some basic economics of crowdfunding admitting that "to economists, the recent rise of crowdfunding, ... which offers little opportunity for careful due diligence, ... is initially startling." Among the other observations, they show that with the introduction of crowdfunding, financing projects is no longer geographically constrained; funding tends to be skewed (on Kickstarter, 1% of all projects raised 36% of funds); propensity to fund increases with accumulated capital, which may lead to herding; and that backers and creators are initially overoptimistic in the outcomes. The authors were also among the

[3] Hörner and Lambert (2016), p. 2.

first to admit the main problem of crowdfunding: the entrepreneurs have more information than the backers, and so "the information asymmetry problem is exacerbated in the case of early-stage ventures raising capital in a lightly regulated environment where funders are remote and have limited opportunity to perform due diligence in person with the creator."[4]

In a following empirical study by the same authors (Agrawal et al., 2015), they discover that "local and distant funders exhibit different funding patterns." In particular, the effect that in a similar study Lin and Viswanathan (2015) call "home bias"—the higher propensity of the projects to be financed locally despite being advertised online to a broader audience,—still exits and is especially strong for the initial stages of funding, indicating that offline social ties still play an important role in the outcomes of crowdfunding campaigns.

Microeconomics of crowdfunding has been studied by few authors so far. Chemla and Tinn (2016) look at the initial crowdfunding process of raising capital as learning about the market demand by observing the funding rates. They admit the presence of moral hazard, but believe there exist platform-level mechanisms like limited campaign lengths and measures to improve transparency to alleviate the negative effects of moral hazard.

Strausz (2016) studies reward-based crowdfunding models from the perspectives of mechanism design and shows that raising capital by pre-selling the products to future consumers has important efficiency aspects that offset the threats of moral hazard. However, he admits that these efficiency gains may not be easily realizable due to incentive problems. He then describes mechanisms that can be implemented to improve the incentives and achieve efficient outcomes at least in some particular cases.

Empirical studies of equity crowdfunding are rare, but slowly start to emerge as the market develops. Ahlers et al. (2015) conduct empirical study of signaling in equity crowdfunding and conclude that "retaining equity and providing more detailed information about risks can be interpreted as effective signals," while social and intellectual capital seem to have little impact on funding outcomes. Hornuf and Neuenkirch (2016) analyze the pricing schemes of 44 equity crowdfunding campaigns conducted in Germany between 2011 and 2014. They discover

[4] Agrawal et al. (2013), p. 20.

that "campaign characteristics, investor sophistication, progress in funding, herding, and stock market volatility influence backers' willingness to pay," that is, all these variables are important signals that determine the success of the crowdfunding campaign at early stages.

There are papers that study the behavior of the investors, in particular, how they reveal their identity and how it affects the crowdfunding process. Burtch et al. (2016), for example, demonstrate that the decision by the backers to hide their identity negatively influences the likelihood to donate of the subsequent contributors and peer-pressures them to hide their own identity as well.

On the other hand, there are studies of the entrepreneurs' behavior in the crowdfunding market. Gerber and Hui (2013) analyze the motives and deterrents to crowdfunding participation among the project creators and supporters. They classify the motivations and deterrents into broader groups based on the questionnaire designed by the authors.

Some other papers that describe the economics of crowdfunding include Belleflamme et al. (2010), who uncover the problems of crowdfunding from the Industrial Organization perspectives, Belleflamme et al. (2015), who describe the economics of crowdfunding portals, Morse (2015), studying peer-to-peer lending, and Meer (2014), who look at donation-based crowdfunding as a model of charitable giving.

1.11. Contribution to the Crowdfunding Literature

The main focus of the theoretical papers on crowdfunding so far has been on studying of the funding stage. It implies looking at crowdfunding from the perspectives of static moral hazard and thus suggesting traditional solutions from the mechanism design and screening literature that are mainly concerned with the development of the optimal menus of contracts. I complement these studies by shifting the focus to the experimentation stage and simplifying the funding stage. I implement the framework of dynamic agency models to show that the incentives become more important *after* the project is funded as lack of transparency and control and the temptation to divert the funds significantly influence ex post incentives and thus affect the ex ante efficiency of crowdfunded projects. As the result, I show, projects tend to be underfunded, have higher expected time to reach success, and typically involve waste of funds.

1.12. Structure of the Rest of the Chapter

The chapter continues with the description of the crowdfunding model. After that, I describe the first best scenario, when the social planner decides how to work on the project. Then, I explain how the equilibrium is different from the efficient solution and characterize some comparative statics. In the end, I analyze the environment in which the project founder commits to submit audited reports of the aggregate expenditures every now and then, after which I conclude.

2. The Model

The crowdfunding model I construct belongs to the class of dynamic experimentation models. It shares some common features and results with basic dynamic experimentation models, but it also has some unique properties. Due to the presence of convex costs, it produces smooth experimentation paths, as opposed to experimentation paths with kinks typical for traditional models. Having an intertemporal budget constraint means there is no need to satisfy budget limits each time period, which results in a solution, which is more flexible.

Broadly, the model is a two-phase game. The first phase consists of the entrepreneur receiving funds from the investors and agreeing to share the project's success with them. The second phase—of the entrepreneur spending funds and effort on the project to try to succeed. There are no actions by investors at the second phase, and at first phase they act as price takers.

2.1. Players

There are two players in this game: the entrepreneur and a mass of investors.

The entrepreneur : the entrepreneur, or the founder, is risk neutral and she is the primary player as she holds full bargaining power at the first phase of the game, when she makes a take-it-or-leave-it offer, and is fully responsible for the outcome of the second phase of the game, when she determines the equilibrium experimentation paths.

The investors : the investors, or backers, are risk-neutral and behave as a single player who can either agree to the offer at phase one or disagree and take the outside option. The investors have no role in the second phase of the game. Such interpretation of investors as a single unity reflects the fact that online backers in crowdfunding have little negotiation power (they can either contribute or not) and almost no possibility to coordinate or play strategically against other backers due to distances that separate them and small sizes of their contributions.

2.2. Game Structure and Actions

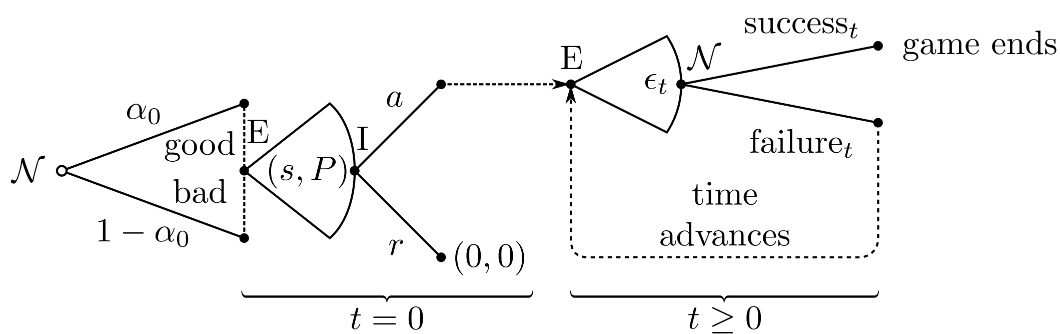
The game is pictured in Figure I.2. The entrepreneur (E) has a project, or venture, that she wants to work on, but does not have funds to do so. This is why the entrepreneur solicits funds from the investors (I). Before the game begins, nature ($\mathcal{N}$) determines if the project is good or bad. This move by nature is unobserved by the players, but the players agree that the prior probability of the project being good is α_0. After nature made the initial move, at time $t = 0$, phase one of the game begins.

The entrepreneur publishes information about her funding campaign on a page located online on some crowdfunding portal's website. The page contains three parts which are important for this model:

1. A description of the project. In this model, the description consists of a single component: α_0, probability that the project is good. If the project is bad, it will never succeed no matter how hard the entrepreneur tries. If the project is good, then it *might* succeed eventually if the entrepreneur exerts effort and spends money on experiments.

2. The amount of funds the entrepreneur seeks to raise, or the funding goal, denoted by P.

Figure I.2. Crowdfunding Game Graph



3. The rewards promised to the backers. If the project succeeds, it generates a certain surplus, and the entrepreneur receives share s of it. The rest will be given to the investors.

This public announcement is essentially a take-it-or-leave-it ("fill or kill") offer to the investors. The investors, acting as a single unity, observe the offer and decide whether to fund the project given α_0, s, and P. If they decide to reject the offer (choice r on the graph), the crowdfunding campaign fails, and the game ends with parties receiving zero payoffs. If the investors agree to fund the project (choice a), then the entrepreneur immediately receives funds P, and the second phase of the game begins.

The second phase of the game is devoted to the entrepreneur's attempts to make the project a success. Every time period $t \geqslant 0$, the entrepreneur decides upon experimentation rate ϵ_t. After this, nature makes a move and determines if the project succeeds at time t. If the project is good, then the higher the experimentation rate is, the higher the chances that the project succeeds. If the project is bad, then success never happens, even if ϵ_t is very high. If success does not happen in period t, time advances to the next period.

If at any point in time the project succeeds, then it generates a certain surplus, which parties share according to s, and the game ends. Otherwise, the game continues indefinitely. So the entrepreneur's pure strategy is a collection of maps:

$$\begin{aligned} &\text{Phase 1: } (s, P) \\ &\text{Phase 2: } ((s, P), A) \mapsto (\epsilon_t, t \geqslant 0), \end{aligned}$$

where $A \in \{\mathrm{a}, \mathrm{r}\}$ is the investors' acceptance or rejection. In phase one, there is no history, so the entrepreneur just select her offer (s, P). In phase two, the history consists of the offer in phase one and of the investors' acceptance or rejection of the offer. The investors' pure strategy is

$$\text{Phase 1: } (s, P) \mapsto A.$$

The history for the investors consists only of the entrepreneur's offer.

2.3. Information

There are two degrees of information asymmetry in this game:

Information about the project : neither the entrepreneur, nor investors know if the project is good or bad.

Information about the entrepreneur's actions : only the entrepreneur knows her level of experimentation, her lever of effort, and how much she spends on the project each time period t. To the investors, this information is unobservable, so there is no way to write contracts contingent on experimentation rates, effort level, or project expenditures.

The only publicly observable and verifiable event regarding the project is the event of success. So all possible contracts in this game can only be contingent on success or no success.

Given that the project is uncertain, both parties have beliefs about the project being good. At the beginning, both parties believe that the project is good with probability α_0. This is enough to define the equilibrium for the game, but it is convenient to define the posterior beliefs that the project is good to simplify the problem statement.

The posterior beliefs that the project is good evolve over time according to the Bayes' Rule. Probability that no success is reached by time t is given by

$$\mathbb{P}\left(\text{no success by time } t\right) = \underbrace{1-\alpha_0}_{\text{project is bad}} + \underbrace{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}_{\substack{\text{project is good,} \\ \text{but has been failing}}},$$

or equivalently (see Appendix A),

$$\mathbb{P}\left(\text{no success by time } t\right) = e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}.$$

Notice the similarity with the exponential distribution. In fact, the exponential distribution describes the arrival of success in this game when the project is certain ($\alpha_0 = 1$) and the experimentation rates are constant ($\epsilon_t = \lambda$, for all t). Posterior probability that the project is still good at time t conditional on no success reached so far is

$$\alpha_t \equiv \mathbb{P}\left(\text{project is good} \mid \text{no success by time } t\right) = \frac{\overbrace{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}^{\substack{\text{project is good,} \\ \text{but has been failing}}}}{\underbrace{1-\alpha_0+\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}_{\text{no success by time } t}},$$

and it evolves in time according to

$$\frac{\mathrm{d}\alpha_t}{\mathrm{d}t} \equiv \dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right).$$

Observe that it means that when the experimentation rates ϵ_t are positive, the posterior belief that the project is good will only decrease. The higher the experimentation rates are, the higher the decrease in the posterior belief will be.

The entrepreneur observes experimentation rates ϵ_t perfectly, so the entrepreneur knows α_t at any point in time. The investors do not see ϵ_t at any point in time, they can only expect the entrepreneur to experiment at certain rates and best respond to what they believe the experimentation rates will be when they agree or disagree to the crowdfunding offer. Since the investors have no roles in the second phase of the game, the evolution of posterior belief α_t only matters for the entrepreneur. The investors simply calculate their expected payoffs from accepting the offer at phase one and decide accordingly.

2.4. Payoffs

The players' expected payoffs will depend on the offer made, (s, P); its acceptance or rejection in phase one, $A \in \{\mathrm{a}, \mathrm{r}\}$; on the experimentation rates exerted by the entrepreneur in phase two of the game, $(\epsilon_t, t \geqslant 0)$; and on players' belief that the project is good, α_0. One obvious thing to notice from the description of the game in Figure I.2 is that if the offer is rejected, then both parties get nothing, independent of their other actions:

$$\pi_E\left(\left(s, P\right), \mathrm{r}, \left(\epsilon_t, t \geqslant 0\right)\right) = \pi_I\left(\left(s, P\right), \mathrm{r}, \left(\epsilon_t, t \geqslant 0\right)\right) = 0,$$

where $\pi_E\left(\cdot\right)$ is the entrepreneur's payoff function, and $\pi_I\left(\cdot\right)$ is the investors' payoff function. If the other was accepted, however, the payoffs are a bit more complicated.

Each experiment requires funding (that is why the entrepreneur raises funds in the first place) and effort. Money and effort are complements, so each experimentation session, at every point in time t, requires $\epsilon_t c$ in monetary expenses and $f\left(\epsilon_t\right)$ in effort costs to produce experimentation rate of ϵ_t. These costs are carried out by the entrepreneur, the investor does not observe neither the entrepreneur's expenses, not the resulting experimentation rates. The total cost of experimenting at rate ϵ_t is

$$f(\epsilon_t) + \epsilon_t c.$$

Function $f(\epsilon)$ is strictly increasing, strictly convex for $\epsilon > 0$, and at least twice continuously differentiable. Also, $f(0) = f'(0) = 0$, while $f''(0) > 0$. For convenience, I assume $f'''(x) \geqslant 0$. I can easily combine the monetary and the effort costs into one convex function, say,

$$g(\epsilon_t) = f(\epsilon_t) + \epsilon_t c,$$

to simply account for the total costs of experimentation at time t at rate ϵ_t. However, in order to balance the entrepreneur's budget constraint I need to know her monetary expenses for each time period. This is why I keep the effort and monetary costs separated.

Convex effort costs in continuous time reflect the fact that keeping the experimentation rate high for prolonged time is harder than keeping it low. Also, it indicates that increases in experimentation rates better be spread over longer periods of time rather than shorter. In the end, it guarantees that experimentation rates will be smoothed, without kinks and jumps. One obvious result of having convex costs is the implication that the entrepreneur would be better off sharing the tasks with other people. Potentially, this is an interesting research topic with tradeoffs between the efficiency gains and losses due to unobservability of the actions. However, in this chapter, I assume that the entrepreneur does a unique job and cannot delegate it to anyone else. Therefore, all the costs are incurred solely by the entrepreneur.

If the experiments are conducted at a positive rate and the project is good, it may eventually succeed at some random finite time T and produce the surplus of size R. The event of success is public and there is no way for the entrepreneur to hide it. If the project is bad, $T = \infty$, and the surplus is never produced.

Future benefits and costs are discounted at rate r, and the rate is common for the parties. The entrepreneur has an opportunity to save funds every period at rate r, which is the same as the common discount rate. Thus the entrepreneur can freely reallocate funds between different time periods without any utility effects. The entrepreneur's expected payoff valued at time $t = 0$ if her offer is accepted is

$$\pi_E\left(\left(s,P\right),\mathrm{a},\left(\epsilon_t,t\geqslant 0\right)\right)=P+\mathbb{E}\left[e^{-rT}sR-\int_0^T e^{-rt}\left(f\left(\epsilon_t\right)+\epsilon_t c\right)\,\mathrm{d}t\right],$$

where the expectation is taken over random success time T. The investors' payoff is simpler:

$$\pi_I\left(\left(s,P\right),\mathrm{a},\left(\epsilon_t,t\geqslant 0\right)\right)=\mathbb{E}\left[e^{-rT}\left(1-s\right)R\right]-P,$$

since they do not need to exert any effort to receive benefits from the project.

Equivalently (see Appendix B), the payoffs of the parties at $t=0$ can be expressed as

$$\pi_E\left(\left(s,P\right),\mathrm{a},\left(\epsilon_t,t\geqslant 0\right)\right)=P+\int_0^\infty e^{-tr-\int_0^t\alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\left[\alpha_t\epsilon_t sR-f\left(\epsilon_t\right)-\epsilon_t c\right]\,\mathrm{d}t,$$

$$\pi_I\left(\left(s,P\right),\mathrm{a},\left(\epsilon_t,t\geqslant 0\right)\right)=\int_0^\infty e^{-tr-\int_0^t\alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\alpha_t\epsilon_t\left(1-s\right)R\,\mathrm{d}t-P.$$

The model works best for the equity-based crowdfunding; the interpretation of s then is straightforward: it's just a share of the surplus the entrepreneur retains while giving the rest to the backers. However, if share $\left(1-s\right)R$ is the investors' share times the surplus the project is expected to generate, then this whole term can have an interpretation of the total value of rewards the backers will receive upon the project success. Alternatively, $\left(1-s\right)R$ can be treated as the amount of debt plus percentages the peer-to-peer lenders will get when the project succeeds (assuming the entrepreneur will not be able to return the debt if she fails completing the project). Thus this model can apply to different forms of crowdfunding, not only to the equity-based model.

2.5. Equilibrium Concept

The equilibrium concept is Sequential Equilibrium in pure strategies. The common prior that the project is good, α_0, does not depend on the game paths, it remains a prior on and off equilibrium. The game is of imperfect information, where Nature moves first and determines if the project is good or bad. This move is unobserved by the Entrepreneur and the Investors, but since Nature is not a strategic player and always selects the project as "good" with probability α_0 and "bad" with probability $1-\alpha_0$, then along all possible equilibria paths and also off the equilibria paths, players have the same beliefs regarding this Nature's move, and these beliefs are consistent.

To satisfy the requirement of sequential rationality, the game is solved using backward induction. At phase two of the game, when the offer is accepted, the entrepreneur determines optimal sequence of experimentation rates, $(\epsilon_t, t \geqslant 0)$ believing that the project is good with probability α_0. At phase one, expecting the entrepreneur to behave rationally in the future, the investors observe the offer and decide if they want to accept or reject it based on their belief α_0 and their expected payoffs in case of acceptance. Finally, knowing which offers the investors will accept and which reject, and how she will behave in case of acceptance, the entrepreneur makes the take-it-or-leave-it offer that is the most beneficial to her.

2.6. Key Assumptions

The goal of my analysis is to reveal why crowdfunding is inefficient and how to deal with inefficiency. I ignore some properties of crowdfunding markets in order to concentrate on this goal. Some important limitations of the model are:

Transaction costs are negligible : this assumption is typical for most economic models. Here, it implies that there are no costs of writing a contract and transferring funds. However, some transaction costs are built in the model: it is prohibitively expensive to monitor the entrepreneur's actions and it is impossible for any party to learn if the project is good or bad. It is also very costly for the investors to coordinate and play strategically against the entrepreneur to improve their bargaining outcomes. Finally, complex conditional contracts are out of reach as well, due to the limitations of the current crowdfunding practices.

Crowdfunding portals are passive : the role of the crowdfunding portal in this model is passive. I assume that portals do not receive any benefits from the interaction between the entrepreneur and the investors, and they do not play strategically. In actual crowdfunding environments, portals receive fees from every successful campaign, and they may influence the funding process by removing the projects from their web sites, by blocking funding, or in many other ways. However, I believe their role is limited and for the purposes of this model, it can be deemed passive and the fees they collect can be ignored. The structure of the interaction between the entrepreneur and the investors reflects the rules set up by the portal.

The funding stage is trivial : the initial stage of collecting the funds and advertising the project is simplified in this model. The purpose of this model is to show that the entrepreneur experiments at inefficient rates. So I ignore all the complexities of the funding phase as it only matters for the experimentation phase if the entrepreneur manages to raise enough funds. The questions of why projects receive enough or not enough funds, what role does herding behavior play in investors' decisions, and how to design the optimal offers are potentially interesting, but out of scope of my research.

3. The Efficient Outcome

3.1. Problem Statement

The efficient solution must be reasonable enough to act as a benchmark, which, at least theoretically, can be reached. This is why the treatment of efficiency in this game entails that even the social planner is uncertain about the project being good or bad. Otherwise, the outcome is trivial: bad projects should not be worked on at all, and good projects should be worked on at a certain constant rate until success happens. This level of efficiency is only achievable if there was some way to know that the project is good, but not knowing that the project is good or bad is not just some flaw of the information distribution between the agents. In this model, it is a fundamental property of the project, and the only way to find out if the project is good is to work on it and achieve success.

The utility functions are transferable in this game, so to produce the efficient outcome, it is enough to combine the expected utility functions and maximize the resulting "social" expected utility. Effectively it means that the contracting phase of the game, or phase one, becomes irrelevant as instead of two entities trying to agree on transfers and shares, we just have one who has the project and the funds to finance it. Therefore, the problem boils down to determining the efficient experimentation path, $(\epsilon_t^*, t \geqslant 0)$.

The combined expected utility is

$$\begin{aligned}\pi\left((s, P), \mathrm{a}, (\epsilon_t, t \geqslant 0)\right) = P &+ \int_0^\infty e^{-tr - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left[\alpha_t \epsilon_t s R - f\left(\epsilon_t\right) - \epsilon_t c\right] \mathrm{d}t \\ &+ \int_0^\infty e^{-tr - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \alpha_t \epsilon_t \left(1 - s\right) R \, \mathrm{d}t - P\end{aligned}$$

$$= \int_0^\infty e^{-tr - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left[\alpha_t \epsilon_t R - f(\epsilon_t) - \epsilon_t c\right] \, \mathrm{d}t.$$

To write the problem of finding the efficient solution in the form of a control problem, I need the expression inside the integral to be of form

$$I(x_t, u_t, t),$$

where x_t is a vector of state variables, and u_t is a vector of control variables. I only have one control variable, ϵ_t for every period t, but the state vector is more complicated. One of the state variables is the posterior belief that the project is good, α_t. Another state variable is the probability that the project will succeed in the future, at random time T (including mass at infinity),

$$M_t = \mathbb{P}(T > t) = e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}.$$

The trajectory of the state variables is characterized by equations of motion that give the time rate of change for each state variable:

$$\begin{aligned} \dot{\alpha}_t &\equiv \frac{\mathrm{d}\alpha_t}{\mathrm{d}t} = -\alpha_t \epsilon_t (1 - \alpha_t), \\ \dot{M}_t &\equiv \frac{\mathrm{d}M_t}{\mathrm{d}t} = -\alpha_t \epsilon_t M_t. \end{aligned}$$

Notice that both equations of motion do not depend specifically on time t, so both are autonomous.

Thus the problem of finding the efficient experimentation path is a dynamic control problem:

$$\begin{aligned} &\max_{(\epsilon_t, t \geqslant 0)} \int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t R - f(\epsilon_t) - \epsilon_t c\right] \, \mathrm{d}t \\ &\text{subject to: } \dot{\alpha}_t = -\alpha_t \epsilon_t (1 - \alpha_t), \\ &\dot{M}_t = -\alpha_t \epsilon_t M_t, \\ &\alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\ &M_0 = 1, \\ &\epsilon_t \geqslant 0, \forall t \geqslant 0. \end{aligned} \tag{I.1}$$

When this problem is solved, the social planner will determine s and P that satisfy:

$$P + \int_0^\infty e^{-tr - \int_0^t \alpha_\tau \epsilon_\tau^* \, \mathrm{d}\tau} \left[\alpha_t \epsilon_t^* s R - f(\epsilon_t^*) - \epsilon_t^* c\right] \, \mathrm{d}t \geqslant 0,$$

$$\int_0^{\infty} e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^* \,\mathrm{d}\tau} \alpha_t \epsilon_t^* (1-s) R \,\mathrm{d}t - P \geqslant 0,$$

so that both parties will be better off participating. This is just a matter of determining transfer sizes and it is not an issue as long as there is a positive expected surplus to gain. Because of this, I only concentrate on characterizing the efficient experimentation path.

3.2. The Solution

The solution to the control problem exists (see Appendix I.B.1), it is unique, stationary (does not depend on time explicitly), and it is characterized by differential equation

$$r\left[\alpha R - f'\left(\epsilon^*\left(\alpha\right)\right) - c\right] = \alpha\left[f'\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right) - f\left(\epsilon^*\left(\alpha\right)\right) + \left(1-\alpha\right) f''\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right)\epsilon^{*\prime}\left(\alpha\right)\right]$$

together with boundary condition

$$\epsilon^*\left(\frac{c}{R}\right) = 0.$$

There is a simple interpretation for the boundary condition. Since α is the posterior belief that the project is good, then $\frac{c}{R} = \underline{\alpha}$ is the lower bound on the belief level. If $\alpha > \underline{\alpha}$, then it may be reasonable to experiment. If $\alpha \leqslant \underline{\alpha}$, then the only reasonable thing to do is to abstain from experimentation completely, that is to have $\epsilon^*\left(\alpha\right) = 0$, because the immediate net benefit from experimentation,

$$\alpha\epsilon R - f\left(\epsilon\right) - \epsilon c < \frac{c}{R}\epsilon R - f\left(\epsilon\right) - \epsilon c = -f\left(\epsilon\right)$$

will be negative, for any $\alpha < \frac{c}{R}$ and $\epsilon > 0$. Since the posterior belief that the project is good can only decrease in time, then having positive experimentation rates in any time period after the belief level of $\frac{c}{R}$ is reached will be wasteful. Alternatively, α can be a measure of the optimism level, with $\alpha = 0$ indicating no confidence in the project, $\alpha = 1$—complete confidence, and some less-than-complete confidence in between.

3.3. Properties of the Efficient Solution

Important properties of the efficient experimentation rate are summarized in Proposition I.1 and pictured in Figure I.3.

Proposition I.1. *The efficient outcome in dynamic crowdfunding models with convex experimentation costs satisfies:*

Some projects are not worth the risk *: if $\alpha_0 \leqslant \frac{c}{R}$, then the project is better left alone.*

Staticity at the top *: the experimentation rate for sure projects ($\alpha = 1$) is stationary.*

The experimentation rate strictly decreases in time *: the policy function strictly increases in α.*

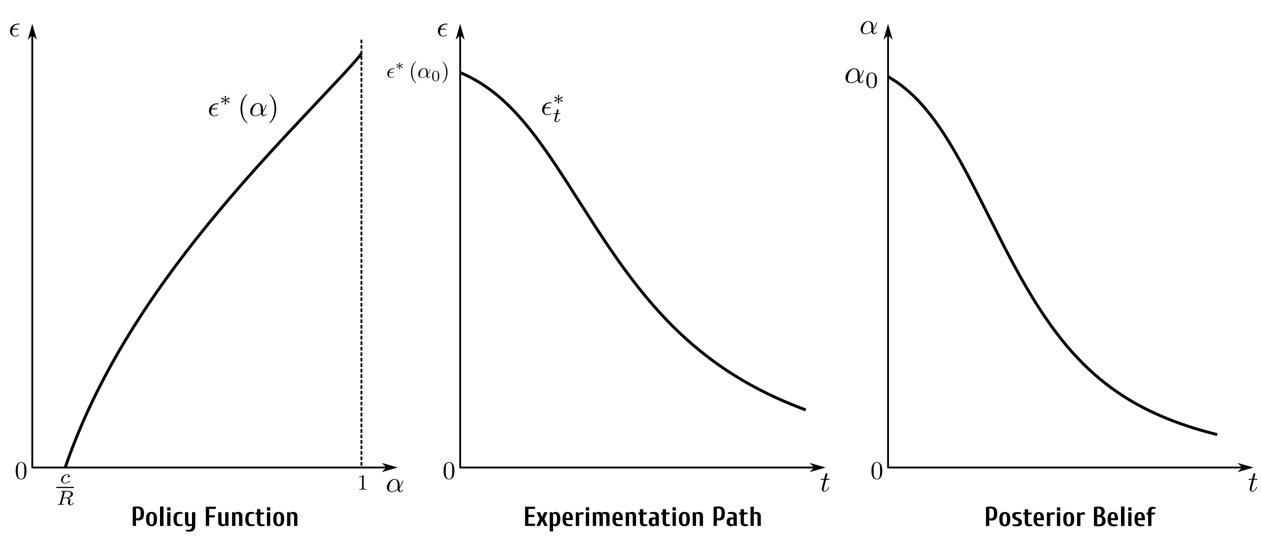
Experimenting never stops *: despite that experimentation rate strictly decreases over time, experiments only stop if success happens, otherwise, experimentation rate is always positive.*

Proof. Complete proof can be found in Appendix I.B.2. I only provide the intuition here:

Some projects are not worth the risk : some projects are so bad that it is unreasonable to spend any effort on them. Suppose it was worth funding the project with prior $\alpha_0 \leqslant \frac{c}{R}$. If the prior belief that the project is good, α_0, is lower than $\frac{c}{R}$, then the posterior α_t can only be lower than that as it decreases over time when no success happens: $\alpha_t \leqslant \alpha_0 \leqslant \frac{c}{R}$. So for every time period t, if $\epsilon_t \geqslant 0$,

$$\alpha_t \epsilon_t R - f(\epsilon_t) - \epsilon_t c \leqslant \frac{c}{R} \epsilon_t R - f(\epsilon_t) - \epsilon_t c = -f(\epsilon_t) \leqslant 0.$$

Figure I.3. The Efficient Outcome



Then combined social payoff can only be negative as

$$\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left[\alpha_t \epsilon_t R - f(\epsilon_t) - \epsilon_t c\right] \mathrm{d}t \leqslant 0.$$

There is a better strategy: conducting no experiments will yield zero. Receiving nothing is better than suffering losses. Therefore, there is no point in trying to make a project with $\alpha_0 \leqslant \frac{c}{R}$ a success. Notice that $\frac{c}{R}$ is also the lower bound on the posterior belief level. It means that if $\alpha_0 > \frac{c}{R}$, there is still an opportunity to make the project a success as initial efficient experimentation rates will be strictly positive.

Staticity at the top : for sure projects, with prior $\alpha_0 = 1 > \frac{c}{R}$, posterior α_t stays fixed no matter what: $\alpha_t = \alpha_0 = 1$. The efficient experimentation rate must satisfy

$$r\left[R - f'(\epsilon^*(1)) - c\right] = f'(\epsilon^*(1))\,\epsilon^*(1) - f(\epsilon^*(1)),$$

which is just a simple equation with the unique solution, given convexity of $f(\cdot)$. Therefore, the experimentation rate stays strictly positive, finite, and constant for every time period if the project is sure to succeed.

Experimentation rate strictly decreases in time : for $\alpha \in \left(\frac{c}{R}, 1\right)$, experimentation rate *increases* in α. Posterior belief α_t strictly decreases over time observing no success, so the efficient experimentation rate, ϵ_t^*, decreases in time.

The proof is by contradiction:

- the efficient experimentation rate is continuously differentiable;
- it is always the case that $\epsilon^*\left(\frac{c}{R}\right) = 0$ and $\epsilon^*(1) > 1$, provided that $\frac{c}{R} < 1$;
- so function $\epsilon^*(\alpha)$ must increase somewhere on $\left(\frac{c}{R}, 1\right)$: see Figure I.3;
- suppose it also decreases on $\left(\frac{c}{R}, 1\right)$, then there must be at least one local maximum in this interval;
- but all the extreme points that can exist on $\left(\frac{c}{R}, 1\right)$ can only be local minima, because having a local maximum is incompatible with the differential equation that characterizes the efficient policy function;

- this is a contradiction to the function decreasing on the interval;
- therefore, function $\epsilon^*(\alpha)$ strictly increases in α on $\left(\frac{c}{R}, 1\right)$.

The efficient experimentation rate increases in posterior belief level α and, consequently, decreases in time.

Experimenting never stops : experiments never stop. For $\alpha \in \left[\frac{c}{R}, 1\right]$, efficient policy function $\epsilon^*(\alpha)$ is strictly increasing and is everywhere below critical function $\bar{\epsilon}(\alpha)$ from Appendix C. The critical function is constructed in such a fashion to ensure that if it is followed, then the experiments stop in finite time. Any function that is strictly below the critical function, but strictly positive for $\alpha \in \left(\frac{c}{R}, 1\right]$, does not provide enough experimentation effort to stop in finite time. Therefore, efficient experimentation should not stop until the project succeeds.

□

4. The Equilibrium

I solve the game by backward induction and show that most forms of crowdfunding are inefficient. Then I describe the solution and some important properties of the equilibrium experimentation path.

4.1. Phase Two of the Game

I begin solving the game from phase two. At the beginning of phase two, the entrepreneur has received the funds, P, and agreed to provide share s of the surplus the project generates if it succeeds to the investors. The investors have no more roles in the game, so it is up to the entrepreneur to decide what to do next.

The entrepreneur is ready to begin experimenting. Suppose that she devises a plan to experiment according to path ϵ_t. Then each time period t, her costs will be

$$f(\epsilon_t) + \epsilon_t c,$$

of which $\epsilon_t c$ is the monetary part. If success happens at time T, then the total amount of funds required to finance the experiments is equal to

$$\int_0^T \epsilon_t c \, \mathrm{d}t.$$

Assume that it is possible to save funds at a rate which is exactly equal to the discount rate r. Then the total amount of funds that the entrepreneur needs to have to successfully finance the experiments up until time T is equal to

$$\int_0^T e^{-rt} \epsilon_t c \, \mathrm{d}t.$$

Unfortunately, the entrepreneur does not know for sure when the project succeeds. So she may theoretically run intro a situation when she still planned experiments in the future, but ran out of money. Therefore, she needs to receive enough funds to finance the project under assumption that it never succeeds. Otherwise, if she expects to run out of money at time T, there is no reason to plan any experiments past time T. She might just assume $\epsilon_t = 0$ for all $t > T$. If it happens that the project succeeds, but she still has some funds left, she may just use the funds for her personal consumption.

Therefore, the main budget constraint at phase two of the game is

$$P \geqslant \int_0^\infty e^{-tr} \epsilon_t c \, \mathrm{d}t,$$

where P is the total amount of funds given to the entrepreneur by the investors. The problem of finding the equilibrium path is then a problem of maximizing the entrepreneur's payoff function given known share s, total amount of funds P, and the budget constraint:

$$\begin{aligned}
&\max_{(\epsilon_t, t \geqslant 0)} \left[P + \int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t s R - f\left(\epsilon_t\right) - \epsilon_t c \right] \, \mathrm{d}t \right] \\
\text{subject to: } & P \geqslant \int_0^\infty e^{-tr} \epsilon_t c \, \mathrm{d}t, \\
& \dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right), \\
& \dot{M}_t = -\alpha_t \epsilon_t M_t, \\
& P, s \text{ given,}
\end{aligned}$$

$$\alpha_0 \text{ given}, \alpha_0 \in [0,1],$$

$$M_0 = 1,$$

$$\epsilon_t \geqslant 0, \forall t \geqslant 0.$$

It is hard to determine if the budget constraint binds in equilibrium, so I need to consider two cases, when it binds and when it does not. Assume, for now, that the budget constraint does not bind (but it is satisfied):

$$P > \int_0^\infty e^{-tr} \epsilon_t c \, \mathrm{d}t.$$

Then P is just a number and does not affect the maximization problem. The problem becomes

$$\max_{(\epsilon_t, t \geqslant 0)} \int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t s R - f(\epsilon_t) - \epsilon_t c\right] \, \mathrm{d}t$$

$$\begin{aligned}
\text{subject to: } & \dot{\alpha}_t = -\alpha_t \epsilon_t (1 - \alpha_t), \\
& \dot{M}_t = -\alpha_t \epsilon_t M_t, \\
& s \text{ given}, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0,1], \\
& M_0 = 1, \\
& \epsilon_t \geqslant 0, \forall t \geqslant 0.
\end{aligned}$$

In this form, the problem looks remarkably similar to the problem of finding the efficient experimentation path, (I.1). The solution must be somewhat similar as well. The only difference is that surplus R is premultiplied by share s, which is just a number at phase two of the game.

Thus, if the budget constraint does not bind at phase two, the solution is characterized by first order differential equation similar to (I.c), but with "sR" instead of just "R":

$$\begin{gathered}
r\left[\alpha s R - f'\left(\epsilon^{**}(\alpha)\right) - c\right] \\
= \alpha\left[f'\left(\epsilon^{**}(\alpha)\right) \epsilon^{**}(\alpha) - f\left(\epsilon^{**}(\alpha)\right) + (1-\alpha) f''\left(\epsilon^{**}(\alpha)\right) \epsilon^{**}(\alpha) \epsilon^{**\prime}(\alpha)\right]
\end{gathered}$$

and boundary condition

$$\epsilon^{**}\left(\frac{c}{sR}\right) = 0.$$

To consider all the possibilities, assume that the budget constraint binds. Then

$$P=\int_0^\infty e^{-tr}\epsilon_t c\,\mathrm{d}t,$$

and the problem can be stated as follows:

$$\begin{aligned}
&\max_{(\epsilon_t,t\geqslant 0),\lambda}\left[\int_0^\infty e^{-tr}M_t\left[\alpha_t\epsilon_t sR-f\left(\epsilon_t\right)-\epsilon_t c\right]\,\mathrm{d}t+\lambda\left(P-\int_0^\infty e^{-tr}\epsilon_t c\,\mathrm{d}t\right)\right]\\
&\text{subject to: } P=\int_0^\infty e^{-tr}\epsilon_t c\,\mathrm{d}t,\\
&\qquad\dot{\alpha}_t=-\alpha_t\epsilon_t\left(1-\alpha_t\right),\\
&\qquad\dot{M}_t=-\alpha_t\epsilon_t M_t,\\
&\qquad P,s\text{ given},\\
&\qquad\alpha_0\text{ given, }\alpha_0\in\left[0,1\right],\\
&\qquad M_0=1,\\
&\qquad\epsilon_t\geqslant 0,\ \forall t\geqslant 0.
\end{aligned}$$

This problem is solved in Appendix I.C.1. The solution in terms of policy function is characterized by first order differential equation

$$\begin{aligned}
&r\left[\alpha sR-f'\left(\epsilon^{**}\left(\alpha\right)\right)-c\left(1+\lambda\frac{1-\alpha}{1-\alpha_0}\right)\right]\\
&=\alpha\left[f'\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)-f\left(\epsilon^{**}\left(\alpha\right)\right)+\left(1-\alpha\right)f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)\epsilon^{**\prime}\left(\alpha\right)\right].
\end{aligned}$$

and boundary condition

$$\epsilon^{**}\left(\frac{c\left(1-\alpha_0\right)+c\lambda}{sR\left(1-\alpha_0\right)+c\lambda}\right)=0.$$

The only difference between this condition and the one with the non-binding budget constraint is that here, the marginal experimentation cost is multiplied by

$$\left(1+\lambda\frac{1-\alpha}{1-\alpha_0}\right).$$

So when $\lambda\geqslant 0$ (the budget constraint binds) and $\alpha\leqslant\alpha_0$ (posterior belief that the project is good can only be lower than prior α_0), the policy function for the case with the binding constraint is always below than the policy function in the non-binding-constraint case, other things equal (see Appendix I.D.2). Having to satisfy the budget constraint is similar to having the cost of experimentation increasing over time. This is not surprising, and it provides clear interpretation of λ as a cost of satisfying the budget constraint.

4.2. Phase One of the Game

I have the solutions to the second phase of the game for the cases when the budget constraint binds and when it does not. In order to find out if the constraint matters, I derive the solution for the first phase of the game, keeping in mind the results for phase two. I show that the budget constraint binds, but the entrepreneur will want to keep the multiplier associated with the constraint as low as possible.

First, consider the acceptance stage of the game. The investors will accept the entrepreneur's offer of (s, P) only if

$$\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau} \alpha_t \epsilon_t^{**} (1-s)\, R\,\mathrm{d}t - P \geqslant 0,$$

where ϵ_t^{**} is the equilibrium experimentation path produced at phase two of the game depending on s and P. Suppose that

$$\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau} \alpha_t \epsilon_t^{**} (1-s)\, R\,\mathrm{d}t - P > 0.$$

Then for any share s, asking for more P is a good idea. If the budget constraint does not bind, the equilibrium path will not be affected, but the entrepreneur will enjoy having more funds to herself. If the budget constraint binds, having more P will mean more relaxed constraint and, consequently, higher utility. Thus, in equilibrium

$$P = \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau} \alpha_t \epsilon_t^{**} (1-s)\, R\,\mathrm{d}t.$$

As expected, the entrepreneur will make investors break even and indifferent between accepting or rejecting the funding offer.

To find out if the budget constraint matters at phase two of the game, I will check two cases: when it does not bind, and when it does. Suppose that the budget constraint does not bind in the second phase of the game. Then the entrepreneur's problem at phase one is

$$\max_{s,P}\left[P+\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau}\left[\alpha_t\epsilon_t^{**}sR-f\left(\epsilon_t^{**}\right)-\epsilon_t^{**}c\right]\,\mathrm{d}t\right]$$

$$\text{subject to: } P=\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau}\alpha_t\epsilon_t^{**}\left(1-s\right)R\,\mathrm{d}t,$$

$$\dot{\alpha}_t=-\alpha_t\epsilon_t^{**}\left(1-\alpha_t\right),$$

$$\alpha_0 \text{ given, } \alpha_0\in\left[0,1\right],$$

where ϵ_t^{**} is the equilibrium experimentation path calculated at phase two based on share s. Plugging the first constraint directly into the objective function, I produce:

$$\max_{s}\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau}\left[\alpha_t\epsilon_t^{**}R-f\left(\epsilon_t^{**}\right)-\epsilon_t^{**}c\right]\,\mathrm{d}t$$

$$\text{subject to: } \dot{\alpha}_t=-\alpha_t\epsilon_t^{**}\left(1-\alpha_t\right),$$

$$\alpha_0 \text{ given, } \alpha_0\in\left[0,1\right].$$

The objective function is similar to the objective function from the problem of determining the first best experimentation path, (I.1). The only component that depends on s is experimentation path $(\epsilon_t^{**}, t\geqslant 0)$. Without any more constraints, the solution to this problem is straightforward: setting $s=1$ allows the entrepreneur to capture the maximal social surplus and achieve first best. Unfortunately, $s=1$ implies that

$$P=\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau}\alpha_t\epsilon_t^{**}\left(1-s\right)R\,\mathrm{d}t=0,$$

so the entrepreneur will receive no funds from the investors if she does not offer anything in return. This would work if she did not have to ask the investors for funds to begin with. Without funds provided by the investors, she will not be able to experiment. Thus the budget constraint must bind.

When the budget constraint binds at phase two, the entrepreneur's problem at phase one can be expressed as

$$\max_{s,P}\left[P+\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau}\left[\alpha_t\epsilon_t^{**}sR-f\left(\epsilon_t^{**}\right)-\epsilon_t^{**}c\right]\,\mathrm{d}t\right]$$

$$\text{subject to: } P = \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau} \alpha_t \epsilon_t^{**} (1-s) R \,\mathrm{d}t,$$
$$P = \int_0^\infty e^{-tr} \epsilon_t^{**} c \,\mathrm{d}t,$$
$$\dot{\alpha}_t = -\alpha_t \epsilon_t^{**} (1-\alpha_t),$$
$$\alpha_0 \text{ given}, \alpha_0 \in [0,1],$$

or, after substituting P everywhere by the first constraint, as

$$\begin{aligned} &\max_s \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau} \left[\alpha_t \epsilon_t^{**} R - f\left(\epsilon_t^{**}\right) - \epsilon_t^{**} c\right] \,\mathrm{d}t \\ &\text{subject to: } \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\,\mathrm{d}\tau} \alpha_t \epsilon_t^{**} (1-s) R \,\mathrm{d}t = \int_0^\infty e^{-tr} \epsilon_t^{**} c \,\mathrm{d}t, \\ &\dot{\alpha}_t = -\alpha_t \epsilon_t^{**} (1-\alpha_t), \\ &\alpha_0 \text{ given}, \alpha_0 \in [0,1]. \end{aligned} \tag{I.2}$$

Notice that, as in the case of the unconstrained problem described earlier, the objective function is the same as for the problem of finding the efficient experimentation path. It captures the fact that the entrepreneur receives all the surplus produced as the result of the experimentation. However, this time, share s is included in two components: it affects the equilibrium experimentation path, $(\epsilon_t^{**}, t \geqslant 0)$ and it is included directly in the budget constraint. Therefore, the solution is not as obvious as it was in the case of ignoring the budget constraint. Lemma I.1 establishes that the budget constraint binds at phase two, but the Lagrange multiplier associated with it is equal to zero.

Lemma I.1. *In dynamic crowdfunding models with convex experimentation costs and intertemporal budget constraints, the entrepreneur asks for exactly as much funds as she needs to carry on with experiments even observing no success (budget constraint binds). However, the constraint will not be restrictive in the sense that the entrepreneur will not have to limit her experimentation rates due to the risk of running out of funds (Lagrange multiplier associated with the constraint is zero).*

Proof. I already established above that the budget constraint binds. What is left to show is that in equilibrium the Lagrange multiplier associated with the budget constraint, λ, is zero.

For the sake of contradiction, suppose that $\lambda > 0$. The interpretation of λ as a shadow cost implies that there is a strictly positive marginal utility of relaxing the budget constraint to be realized at phase two. From the perspectives of phase one, it means that the entrepreneur can ask the investors for a little bit more funds in order to improve her utility at phase two. To accommodate this increase in the investments and still keep the constraint,

$$P = \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**} \, \mathrm{d}\tau} \alpha_t \epsilon_t^{**} (1-s) \, R \, \mathrm{d}t,$$

binding, the entrepreneur will have to share part of the *additional* utility produced at phase two with the investors. It is possible because $\lambda > 0$ implies that there are strictly positive gains from relaxing the budget constraint even by a very small amount, so there is always a room for sharing. Practically, it will result in a small reduction of the share, s^{**}, offered. Given that the entrepreneur receives the whole surplus at phase one, she will ultimately capture the whole increase in the surplus by directly benefiting from having a more relaxed budget constraint at phase two and by receiving more funds at phase one.

When $\lambda > 0$, the entrepreneur can improve her utility by asking the investors for more funds, P, in exchange for a slightly lower share, s, for herself. Thus situation when $\lambda > 0$ cannot be an equilibrium, it would be a contradiction. Because of that, in equilibrium, the budget constraint binds at phase two, but it is not restrictive and the Lagrange multiplier, λ, associated with it is zero. □

Therefore, there exists a unique equilibrium share, $s^{**} \in (0, 1)$, that makes the budget constraint bind, but keeps the Lagrange multiplier λ equal to zero. The entrepreneur neither wants to increase s^{**} because it will result in a lower surplus, nor wants to decrease it because then the budget constraint will no longer bind and so it will be optimal to increase the share.

4.3. The Solution

The equilibrium of the game in pure strategies is characterized in a series of equations. Entrepreneur's equilibrium strategy at phase two is

$$((s, P), A) \mapsto (\epsilon_t^{**}, t \geqslant 0),$$

where time path $\left(\epsilon_t^{**}, t \geqslant 0\right)$ is derived from policy function $\epsilon^{**}(\alpha)$ every period by the system of equations:

$$\begin{aligned}
\epsilon_0^{**} &= \epsilon^{**}\left(\alpha_0\right), \\
\epsilon_t^{**} &= \epsilon_0^{**} + \int_0^{\infty} \dot{\epsilon^{**}}_\tau \, \mathrm{d}\tau, \\
\dot{\epsilon^{**}}_t &= \epsilon^{**\prime}\left(\alpha_t\right) \dot{\alpha}_t, \\
\alpha_t &= \alpha_0 + \int_0^{\infty} \dot{\alpha}_\tau \, \mathrm{d}\tau, \\
\dot{\alpha}_t &= -\alpha_t \epsilon^{**}\left(\alpha_t\right)\left(1-\alpha_t\right).
\end{aligned}$$

The policy function, $\epsilon^{**}(\alpha)$, depends on share s and budget P. It is characterized by differential equation

$$\begin{gathered}
r\left[\alpha s R - f'\left(\epsilon^{**}(\alpha)\right) - c\left(1 + \lambda(s, P) \frac{1-\alpha}{1-\alpha_0}\right)\right] \\
= \alpha\left[f'\left(\epsilon^{**}(\alpha)\right) \epsilon^{**}(\alpha) - f\left(\epsilon^{**}(\alpha)\right) + (1-\alpha) f''\left(\epsilon^{**}(\alpha)\right) \epsilon^{**}(\alpha) \epsilon^{**\prime}(\alpha)\right]
\end{gathered}$$

and boundary condition

$$\epsilon^{**}\left(\frac{c\left(1-\alpha_0\right) + c\lambda(s, P)}{sR\left(1-\alpha_0\right) + c\lambda(s, P)}\right) = 0.$$

The boundary condition in this form is not easily interpretable, but the usual reasoning suffices: there exists a belief level after reaching which it is unreasonable to continue with experiments. In this case, this belief level accounts for the shadow costs of having to satisfy the budget constraint and it also includes the prior belief that the project is good. Multiplier $\lambda(s, P)$ ensures that the budget constraint is satisfied. It is positive when it binds.

The equilibrium strategy for investors at phase one of the game is simple. They accept any offer of (s, P) that satisfies

$$\int_0^{\infty} e^{-tr - \int_0^t \alpha_\tau \epsilon_\tau^{**} \, \mathrm{d}\tau} \alpha_t \epsilon_t^{**}(1-s) R \, \mathrm{d}t - P \geqslant 0,$$

and reject all other offers.

Finally, the Entrepreneur's equilibrium strategy at phase one is to offer $\left(s^{**}, P^{**}\right)$ such that

$$\lambda\left(s^{**}, P^{**}\right) = 0,$$

$$
\begin{aligned}
s^{**} &= 1 - \frac{\int_0^\infty e^{-tr} \epsilon_t^{**} \, \mathrm{d}t}{\int_0^\infty e^{-tr - \int_0^t \alpha_\tau \epsilon_\tau^{**} \, \mathrm{d}\tau} \alpha_t \epsilon_t^{**} \, \mathrm{d}t} \frac{c}{R}, \\
P^{**} &= \int_0^\infty e^{-tr} \epsilon_t^{**} c \, \mathrm{d}t,
\end{aligned}
$$

The first condition means that the budget constraint will be satisfied on the boundary of being binding, so that no extra waste happens due to shadow costs of satisfying the constraint. The second condition guaranties that the budget constraint is satisfied and the entrepreneur asks for exactly as much as the investors are willing to provide. The third condition shows that the entrepreneur will ask for exactly the amount she needs to have in order to carry on with the experiments even if the project keeps yielding no success over time.

Given that $\lambda \left(s^{**}, P^{**}\right) = 0$, along the equilibrium path, the policy function is determined by differential equation

$$
\begin{gathered}
r \left[\alpha s^{**} R - f' \left(\epsilon^{**} \left(\alpha\right)\right) - c\right] = \\
\alpha \left[f' \left(\epsilon^{**} \left(\alpha\right)\right) \epsilon^{**} \left(\alpha\right) - f \left(\epsilon^{**} \left(\alpha\right)\right) + \left(1 - \alpha\right) f'' \left(\epsilon^{**} \left(\alpha\right)\right) \epsilon^{**} \left(\alpha\right) \epsilon^{**\prime} \left(\alpha\right)\right],
\end{gathered}
\tag{I.3}
$$

and boundary condition

$$
\epsilon^{**} \left(\frac{c}{sR}\right) = 0,
$$

which together characterize the unique experimentation rate for every posterior belief level α.

4.4. Properties of the Equilibrium

The important properties of the equilibrium experimentation rate are summarized in Proposition I.2 and pictured in Figure I.4.

Proposition I.2. *In dynamic crowdfunding models with convex experimentation costs, the equilibrium outcome satisfies:*

Inefficiency *: the equilibrium policy function, $\epsilon^{**} \left(\alpha\right)$, is everywhere below efficient policy function $\epsilon^{*} \left(\alpha\right)$.*

Wastefulness *: the entrepreneur receives more funds than she is expected to spend on the project.*

Some projects are not worth the risk *: if $\alpha_0 \leqslant \frac{2c}{R}$, then the project will not be funded.*

Staticity at the top *: sure projects ($\alpha = 1$) are worked on with the constant rate.*

The experimentation rate strictly decreases in time *: the equilibrium policy function strictly increases in α.*

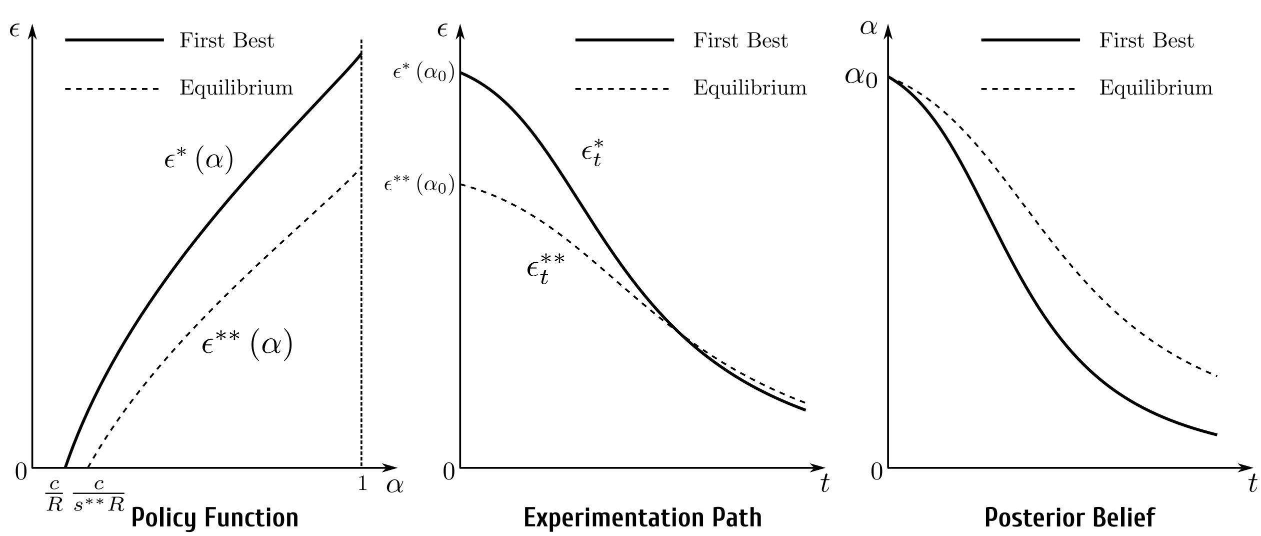
Experimenting never stops *: the experimentation rate is always positive.*

Proof. Most of the important steps of the proof have been described during the description of the equilibrium path. I rely on these intermediate steps here.

Inefficiency : inefficiency is the direct result of the equilibrium policy function being lower than the efficient policy function.

Observe that the only difference between (I.3) with the corresponding boundary condition and (I.c) with its boundary condition is that surplus R is multiplied by s^{**} for the equilibrium outcome. Given that $s^{**} < 1$, $s^{**}R < R$, and thus the equilibrium policy function is everywhere lower than the efficient policy function (see Appendix I.D.2 for the effects of a change in R on the policy function). Appendix I.D.1 establishes that if one policy function is everywhere lower than the other policy function then it generates lower total expected revenue. Therefore, the total

Figure I.4. The Equilibrium and the Efficient Outcome



expected revenue from experimentation in equilibrium is always lower than the efficient total expected revenue. Consequently, the total expected surplus must be lower as well.

Notice that inefficiency does not imply that equilibrium experimentation *rate* is always lower than the efficient experimentation rate *in time* (refer to the middle picture in Figure I.4). This is due to the fact that in equilibrium the posterior belief that the project is good evolves more slowly than in the efficient case (rightmost picture in Figure I.4) and so at some point in time, Entrepreneur will be more optimistic than she should be in the efficient case and thus her experimentation rate may be higher. This indicates that experimentation rates at particular points in time may not be very informative for a crowdfunding analyst.

Wastefulness : this is the inevitable consequence of receiving funds upfront. The entrepreneur needs to ensure that she can carry on with experiments even if she continues to see no success. She requests

$$P^{**} = \int_0^\infty e^{-rt} \epsilon_t^{**} c \, \mathrm{d}t,$$

but she only expects to actually spend

$$\int_0^\infty e^{-rt - \int_0^t \alpha_\tau \epsilon_\tau^{**} \, \mathrm{d}\tau} \epsilon_t^{**} c \, \mathrm{d}t$$

because success can happen at any point in time. When success happens, she will still have funds left:

$$\int_0^\infty e^{-rt} \epsilon_t^{**} c \, \mathrm{d}t > \int_0^\infty e^{-rt - \int_0^t \alpha_\tau \epsilon_\tau^{**} \, \mathrm{d}\tau} \epsilon_t^{**} c \, \mathrm{d}t;$$

and she will be free to use them any way she pleases. This transfer of extra funds is not Pareto decreasing per se in this game, but it lowers the incentives of investors to fund the project, which hampers the efficiency of experimentation. Unfortunately, the effect of receiving funds conditional on no success may be a bad idea because it might create incentives to delay succeeding in experiments in order to receive funds in the future. In any case, crowdfunding with upfront payments is wasteful.

Some projects are not worth the risk : If the prior α_0 is lower than $\frac{2c}{R}$, then it is not worth for the entrepreneur to experiment and it is not worth for the investors to fund the project.

For the project to be worked on, it must be the case that:

$$\alpha_t \epsilon_t s R - f(\epsilon_t) - \epsilon_t c \geqslant 0.$$

So it must be the case that (see proof of Proposition I.1)

$$\alpha_0 \geqslant \alpha_t \geqslant \frac{c}{sR}.$$

For the project to be funded, it must be worth it to fund at least the very first experimenting session:

$$\alpha_0 \epsilon_0 (1-s) R - \epsilon_0 c \geqslant 0,$$

or

$$\alpha_0 (1-s) R - c \geqslant 0,$$
$$\alpha_0 \geqslant \frac{c}{(1-s) R}.$$

All the subsequent experimentation sessions will only promise lower difference between marginal expected revenue $\alpha_t (1-s) R$ and marginal monetary costs c because α_t only decreases in time.

Thus, it must be the case that, both conditions,

$$\alpha_0 \geqslant \frac{c}{sR}, \text{ and}$$
$$\alpha_0 \geqslant \frac{c}{(1-s) R},$$

are satisfied. The lowest α_0 that ensures both these conditions apply is $\alpha_0 = \frac{2c}{R}$. Notice that this threshold is twice the amount of critical belief for the efficient projects. It means that projects with $\frac{c}{R} < \alpha_0 \leqslant \frac{2c}{R}$ will not be crowdfunded despite that they have the potential to generate surplus from the perspectives of Pareto efficiency.

The other properties : the equilibrium policy function defined by (I.3) with its boundary condition is similar to the efficient policy function defined by (I.c) with the corresponding boundary condition. The only difference between the two is that project revenue R is multiplied by share s^{**} for the equilibrium policy function. It does not change the general properties of the experimentation path. Refer to the proof of Proposition I.1 to see why the experimentation rate will be static at the top, strictly decreasing in time, and why experimentation rate will stay positive until success happens, or forever.

□

4.5. Efficient Crowdfunding

I established that crowdfunding models in which funding is received upfront in exchange for the share of the surplus or other benefits conditional on the project success are inefficient. Does it mean that all forms of crowdfunding are inefficient? I argue that some forms of crowdfunding may be theoretically efficient. In particular, pure donation-based crowdfunding can be said to be efficient.

Investment-based crowdfunding is inefficient in general. This is due to the fact that backers, or investors, expect to receive benefits conditional on the project success. This is the primary reason they fund the project, the project founder will have to share part of the surplus with the backers. As demonstrated above, this implies that the experiments will be carried out inefficiently because the entrepreneur's ex post incentives to experiment will be lower than in the efficient case. Donation-based crowdfunding that promises conditional rewards is inefficient for the same reasons: a part of the surplus will be spent to provide benefits to the backers lowering the entrepreneur's incentives to experiment.

If the rewards are unconditional (provided independent of the project success or failure), however, or if the backers are willing to fund the project without expecting anything in return, then crowdfunding can be theoretically efficient. There still may be some misallocation of funds in practice, of course, but within the described model, such forms of crowdfunding will be efficient because the entrepreneur will be the sole recipient of the surplus generated by the project. The entrepreneur's individual incentives to experiment will be closer to the social incentives to maximize the total surplus.

Additionally, allowing the backers who are willing to donate funds for the sake of developing the project without expecting anything in return is beneficial since they must receive some utility out of just participating in the project. The same is true for the unconditional rewards-based crowdfunding. If the investors are willing to pay for the rewards more than it costs the entrepreneur to provide these rewards, then it is beneficial for the parties to trade regardless of the project success.

Therefore, some forms of crowdfunding are efficient in theory. For example, most of the projects on the GoFundMe platform belong to the pure donation-based category because they involve individuals collecting money for personal or charitable projects without offering conditional benefits. This form of crowdfunding is welfare-enhancing, but it is also prone to fraud due to the fact that backers are not interested in the actual outcomes of the projects, they just receive participation prizes (see Fredman, 2015). These types of crowdfunding projects have their own set of problems despite being potentially efficient, but analyzing them is behind the scope of this chapter as I concentrate on crowdfunding models that promise backers conditional benefits in return for their donations.

5. Comparative Statics

It is important to understand how the parameters of the model affect the equilibrium and efficient outcomes. It allows predicting which crowdfunding projects will be worked on more intensely

Table I.2. Model Parameters

Parameter	Description
R	sharable part of the surplus the project generates upon success
c	marginal monetary cost of experimentation
r	discount coefficient, the measure of impatience
α_0	prior belief that the project is good

Table I.3. Effects of the Parameter Changes

Change	s^{**}	P^{**}	Expected Surplus
Baseline	0.59	7.03	14.63
$R\uparrow$	0.69	8.33	23.63
$c\uparrow$	0.42	9.21	11.94
$r\uparrow$	0.60	6.76	14.33
$\alpha_0\downarrow$	0.63	2.98	5.72

and which projects will be farther from efficiency. The main parameters of the model are described in Table I.2. They include the total surplus the project generates upon success, R, the marginal monetary cost of experimentation, c, the discount coefficient, r, and the prior belief that the project is good, α_0. These parameters affect the first best and the equilibrium results through the changes in policy functions, which, in their turn, result in changes in the experimentation paths, the posterior beliefs, and the total surplus produced by the interaction between investors and the entrepreneur.

Appendix I.D.2 contains the argumentation for why certain changes in parameter values result in certain movements of the policy functions. Here, I summarize the directions of these effects and show the results on the graphs. The effects are calculated based on a particular baseline model, but they are illustrative for the general case as the direction of the change does not depend (with the exception of the change in the prior beliefs that the project is good) on the choice of the effort function or parameter values.

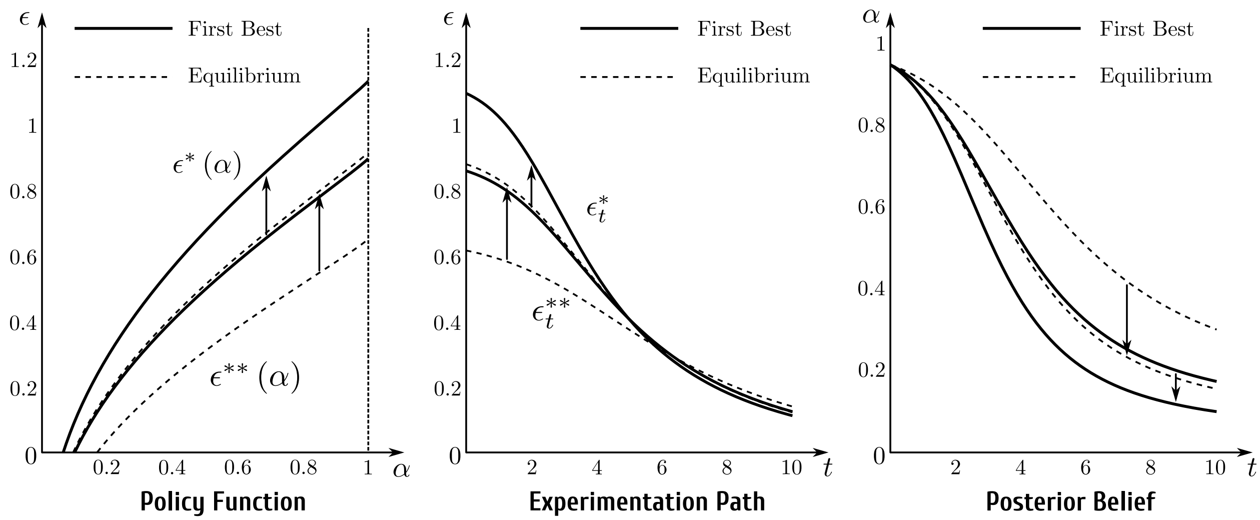
I assume that the effort cost function is $f(x) = x^2$ and the initial parameter values are:

$$R = 20, \qquad c = 2, \qquad r = 0.05, \qquad \alpha_0 = 0.95.$$

Differential equations are solved numerically and graphs are created using Python scripts[5].

[5] Source files are available upon request.

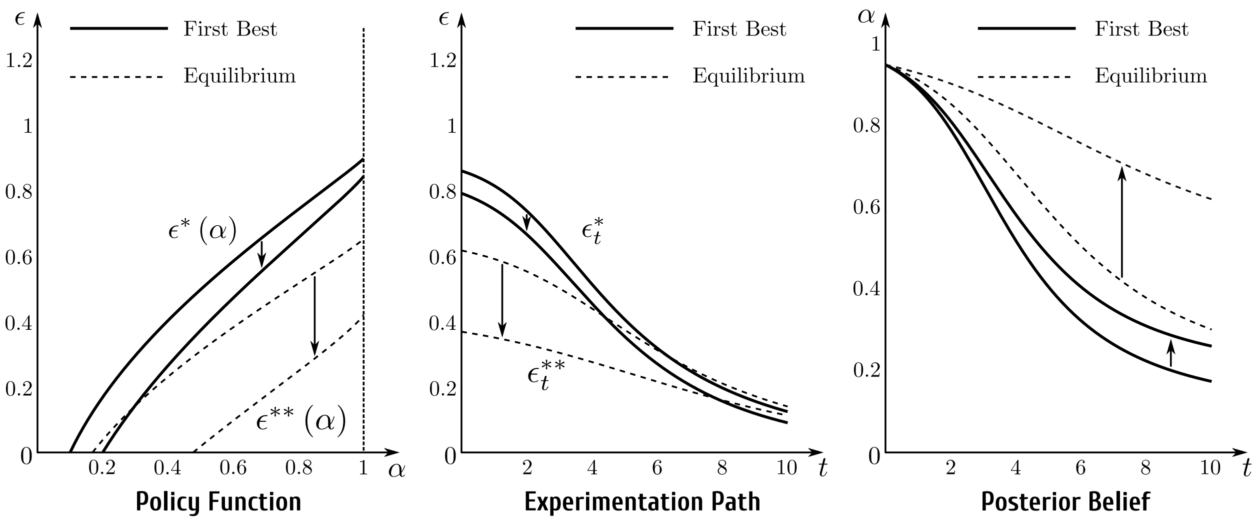
Figure I.5. Effects of an Increase in the Project Surplus



The results of parameter values adjustments are summarized in Table I.3. The baseline parameters result in the total surplus of size 14.63, the investors donate a total of 7.03 monetary units and this is exactly how much they expect to receive in return agreeing that the entrepreneur keeps 59% of the surplus of 20 generated in the event of the project success. Given the high prior of 95%, patient players ($r = 0.05$), and low monetary experimentation costs of 2, this project is expected to be quite successful. The total expected surplus of 14.63 is not that far from the total surplus of 20 the project might generate in the event of success.

First, consider an increase in the surplus the project generates upon success, R. In practice, it means that the project will be expected to be more profitable than prior to the change. Suppose that R increases from 20 to 30 monetary units. The effect of this change on the experimentation rates and the posterior belief evolution is pictured in Figure I.5. An increase in R results in both policy functions, the efficient policy function and the equilibrium function, moving upward (pictured on the left hand side of the figure). Both efficient and equilibrium experimentation rates will be higher for each belief level. Consequently, posterior beliefs will evolve faster, and thus the posterior evolution paths shift downward (pictured on the right hand side): the parties will become pessimistic about the project faster because the entrepreneur will devote higher effort

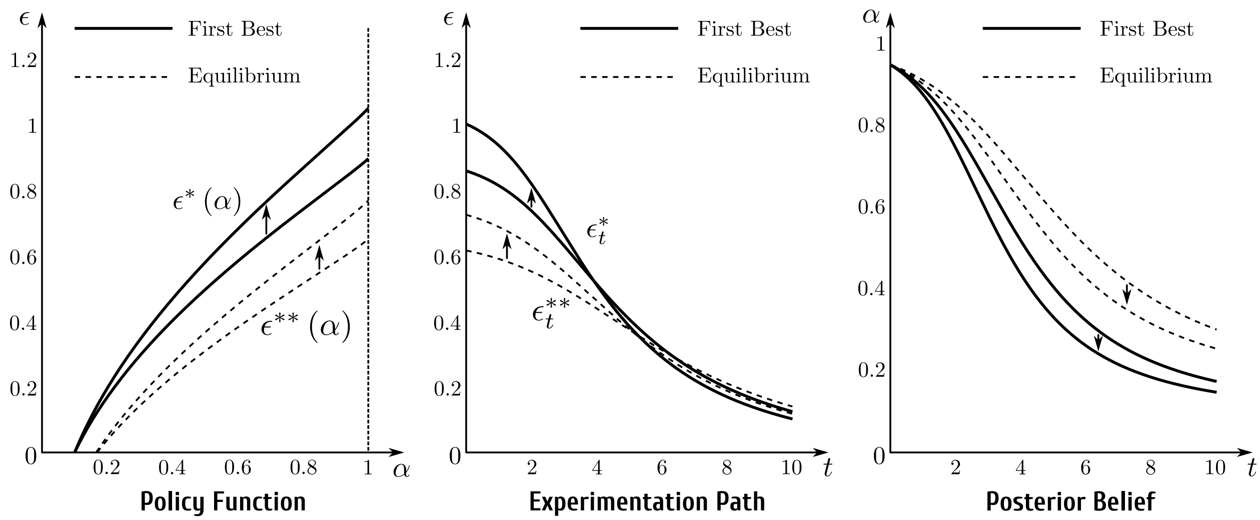
Figure I.6. Effects of a Hike in Marginal Monetary Cost of Experimentation



to completing the project. As expected, equilibrium share s^{**} increases: from 59% to 69%. The amount of funds the entrepreneur requests at equilibrium rises from about 7.03 to 8.33, and the total expected equilibrium surplus increases from 14.63 to 23.63. Overall, the experimentation rates will get higher at early stages of the project development, but since the pessimism now settles faster, after some time, the experimentation rates may become lower than before the change (middle part of the graph).

Second, consider a hike in the marginal monetary cost of experimentation, c, from 2 to 4 monetary units. It means that each experimentation session will be more expensive, and increasing experimentation rates will cost extra. The effect it causes on the efficient and equilibrium policy functions is almost exactly the opposite of the effect of increasing R. It is pictured in Figure I.6. The efficient policy function and the equilibrium policy function, both, move downward (pictured on the left hand side). This results in lower experimentation rates for every belief level, which means that parties become pessimistic about the project slower: belief evolution paths rotate upward (pictured on the right hand side of the Figure). Equilibrium share s^{**} decreases to 42% and the amount of funds requested rises to 9.21 monetary units. The total expected surplus generated by the project drops to 11.94. Effectively, the experimentation rates decrease at early stages, but

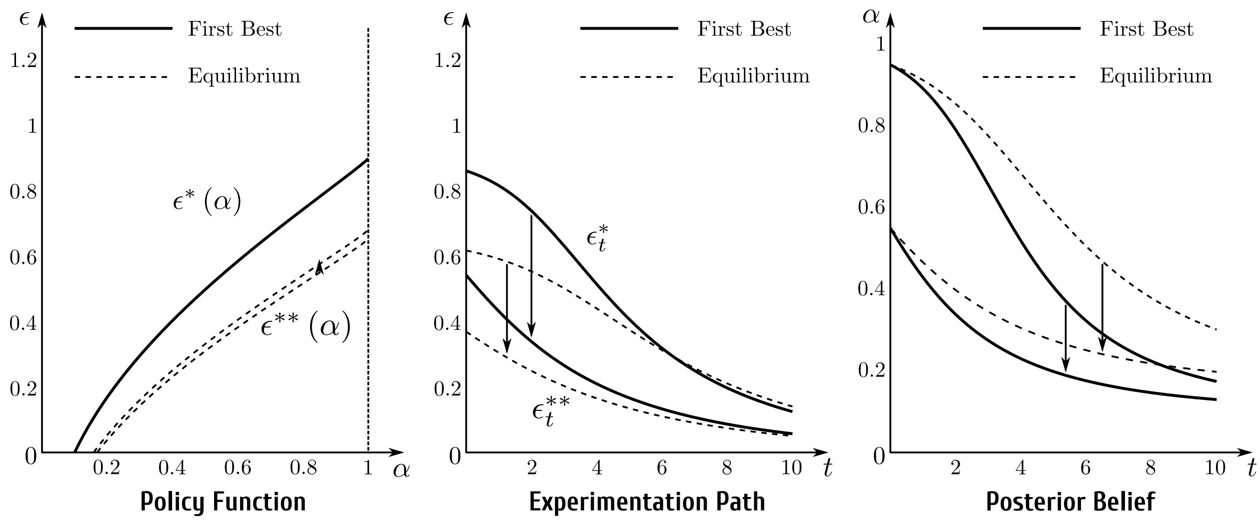
Figure I.7. Effects of a Raise of Discount Rates



given that the parties stay optimistic longer, the experimentation rates increase later in time as compared to the situation with lower marginal monetary cost (the middle part of the graph).

Third, suppose that a discount rate changes from $r = 0.05$ to 0.07. It will indicate that the parties will be relatively less patient, and so they will be less likely to wait for the project to succeed in the future and will be more likely to want to exert higher effort now. The effect of this change on the experimentation rates and beliefs is demonstrated in Figure I.7. The efficient policy function rotates counter-clockwise around point $\left(\frac{c}{R}, 0\right)$. Thus, after the change, Entrepreneur wants to experiment more for higher belief levels, but this desire contracts as she gets more pessimistic. The equilibrium policy function behaves differently: it also rotates counter-clockwise, but it also moves slightly upward indicating that the equilibrium share s^{**} increases and so point $\left(\frac{c}{s^{**}R}, 0\right)$ moves to the left. These effects can be seen on the left hand side of the graph. Otherwise, the result is similar to the effect of an increase in surplus R: the beliefs evolve faster (right hand side of the figure), experimentation rates increase at the early stages and decrease at the later stages as compared to the case with the more patient players. Equilibrium share s^{**} slightly increases to 60%. The amount of requested funds decreases to 6.76. Despite that experimentation rates are now higher for every belief level, the total expected surplus actually declines to 14.33. Thus

Figure I.8. Effects of a Decrease in the Prior Belief that the Project is Good



having lower patience causes the parties to work harder, but expect to receive less because the success still happens in the future, and the parties are less willing to wait.

Finally, suppose that the entrepreneur and the investors are not as optimistic about the perspectives of the project at the beginning of the game: instead of believing the project to be good with probability $\alpha_0 = 95\%$, they believe it to be good with probability of only $\alpha_0 = 55\%$. The effects of this change are demonstrated in Figure I.8. The efficient policy function does not change because it is stationary. The equilibrium policy function only moves because the equilibrium share changes. The direction of the change is unpredictable, but the magnitude is expected to be small. Since the parties begin as less optimistic, the belief evolution paths and the experimentation paths shift downward. Equilibrium share s^{**} increases to 63% in this example, but given different baseline model, it might decreases as the effect is undetermined in general. The amount of funds requested and received drops to 2.98 and the total expected surplus declines to 5.72. The project becomes riskier, thus the parties' expectations about the surplus and the amount of funds they are ready to invest in it decrease.

Therefore, higher surplus R, lower marginal monetary costs of experimentation c, higher patience, or higher optimism level will result in projects with bigger expected total surplus. Projects with higher surplus, lower marginal monetary experimentation costs, and participants with lower patience levels will produce higher experimentation rates. It must be noted that the baseline parameters presented here are pretty favorable, and thus the project was not that far from being efficient. Projects with lower benefits to experimentation costs ratios and project with lower priors will result in experimentation paths which will be much farther from the efficient paths.

6. Reporting, Audit, and Efficiency

The model described so far explained why most forms of crowdfunding are inefficient. The main cause of the inefficiency is the inability of the entrepreneur to commit ex ante to the efficient experimentation path due to low ex post incentives to experiment. After the entrepreneur receives the funds from the investors, her remaining incentives to experiment depend only on the share of the project that she kept for herself. Since the share is lower than one, the entrepreneur is reluctant to exert efficient effort levels. There exists no easy contractual fixes to inefficiency in this model.

If it were a simple one-period model, it would be possible to introduce a contract with financial hostages, for example. Financial hostages are assets of a certain value that will be given by the entrepreneur to the investors at time $t = 0$ and returned to the entrepreneur only in the event of the project success. Carefully calibrating the value of the financial hostages allows to align the entrepreneur's incentives to experiment with the efficient levels: the entrepreneur will be more motivated to experiment because she will want the assets back. Unfortunately, in the dynamic setting of the model I describe, with evolving belief levels and convex effort costs, it will require having the assets with values that evolve over time in a predictable and deterministic way so that the incentives are aligned with the efficient levels at each time frame, t. Not only this requires a well-developed securities market and complex quantitative skills on behalf of the parties, but it is also contractually incompatible with the modern crowdfunding practice. It is simply hard to imagine that the crowdfunding project founders and hundreds of online backers will get together to write and execute a complicated contract that involves financial hostages, let alone financial hostages with the evolving values. A more practical way to improve incentives is to facilitate commitment by making unobservable expenditures partially observable by the means of audited reporting.

I argue that ex post audited disclosure of aggregated experimentation expenditures can improve efficiency. Committing to disclose the amount of money spent on experiments means that the entrepreneur will have to hit certain milestones after she received the funds and be accountable for not reaching the milestones. Given that the entrepreneur receives all the expected surplus in the game, if it allows to improve the overall efficiency, it is beneficial to the entrepreneur. Thus project founders will actually benefit from being accountable.

The entrepreneur will write a simple plan, in which she will outline how much she plans to spend on the project between different dates in the future. Then she will commit to report these expenditures at certain dates and make sure that the reports actually demonstrate the amounts spent on the project. It can be done, in particular, by hiring an auditor to verify the reported amounts and by committing to a certain crippling punishment scheme in case of violations. This way, committing to spend the same amount as in the efficient case will be Pareto improving.

Committing to spend the efficient aggregate amounts of money on experiments is not the only imaginable way to improve efficiency, and it is most probably not the optimal reporting mechanism. However, it is simple and practical and it can work in the constrained contractual environment of crowdfunding, in which writing complex contracts between founders and backers is next to impossible. There are several ways to make it work in practice. The forceful way would be to change the legislation by forcing the crowdfunding project founders to submit regular audited reports of the expenditures on their projects. Another way is to make the crowdfunding portals responsible and accountable for ensuring that the funds transfered to the entrepreneurs are only spent on the projects and not on something personal. An alternative way is to wait for the emergence of the market mechanisms: auditors may decide to offer cost audit services specifically tailored and priced for the needs of the crowdfunding industry.

Looking at the consequences of the JOBS Act from the perspectives that audited reporting improves efficiency, I can conclude that in its current form JOBS Act supports the crowdfunding industry, but it also fosters inefficiency in the market. On the one hand, reporting is costly, so

Table I.4. Reporting Sums

End of Year	Reported Amount
1	$\int_0^1 \epsilon_t c \,\mathrm{d}t$
2	$\int_1^2 \epsilon_t c \,\mathrm{d}t$
3	$\int_2^3 \epsilon_t c \,\mathrm{d}t$
...	...

deregulating the industry and reducing the reporting requirements for startups may have been a good idea as it helped finance the projects that otherwise would have been prohibitively expensive. On the other hand, aiming at reducing the reporting and auditing costs and fostering the emergence of crowdfunding audit market may have been a better idea as it would have allowed the crowdfunding industry to emerge as more efficient and trustworthy. The actual economic impact of the JOBS Act will only be determined as enough data is available to make the empirical analysis possible.

6.1. Phase Two of the Game

Suppose that the reporting requirements exist, reporting is costless, time is in years, and by the end of every year the entrepreneur released the audited reports of the total amount of funds she had spent on experiments over the year. Then the reported amounts will be calculated as in Table I.4. Suppose the entrepreneur commits or is required to commit to spend the efficient amount of funds on the project in aggregate terms every year, that is, it must be the case that, for example, between t and $t+1$,

$$\int_t^{t+1} \epsilon_t c \,\mathrm{d}t = \int_t^{t+1} \epsilon_t^* c \,\mathrm{d}t,$$

where $(\epsilon_t^*, t \geqslant 0)$ is the efficient experimentation path. Simplify by canceling c on both sides:

$$\int_t^{t+1} \epsilon_t \,\mathrm{d}t = \int_t^{t+1} \epsilon_t^* \,\mathrm{d}t,$$

and recall that

$$\alpha_{t+1} = \frac{\alpha_t e^{-\int_t^{t+1} \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_t + \alpha_t e^{-\int_t^{t+1} \epsilon_\tau \, \mathrm{d}\tau}}.$$

Requiring the entrepreneur to spend a certain amount of funds every time period is equivalent to requiring her to reach specific levels of posterior belief α_{t+1} by the end of the period. If at time $t = 0$, the prior is α_0, then at time $t = 1$, along the efficient experimentation path, the posterior will reach

$$\alpha_1 = \frac{\alpha_0 e^{-\int_0^1 \epsilon_\tau^* \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^1 \epsilon_\tau^* \, \mathrm{d}\tau}}.$$

If during the same period the entrepreneur actually conducts experiments at the rate such that

$$\int_0^1 \epsilon_t c \, \mathrm{d}t = \int_0^1 \epsilon_t^* c \, \mathrm{d}t,$$

then by the time when $t = 1$, she would reach the posterior belief level of

$$\frac{\alpha_0 e^{-\int_0^1 \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^1 \epsilon_\tau \, \mathrm{d}\tau}} = \frac{\alpha_0 e^{-\int_0^1 \epsilon_\tau^* \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^1 \epsilon_\tau^* \, \mathrm{d}\tau}} = \alpha_1,$$

which is exactly the same level of belief she would have reached if she had conducted experiments efficiently.

Suppose that the entrepreneur and the investors agreed on a certain share, s, and a funding level, P. Following the logical reasoning similar to the basic equilibrium case described above, I can state that the budget constraint must be satisfied and the multiplier associated with the constraint must be zero. I ignore the constraint at phase two of the game. Given share s, the problem of calculating the experimentation path can be formulated as a maximization problem,

$$J = \max_{(\epsilon_t, t \geqslant 0)} \left[P + \int_0^\infty e^{-rt - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left(\alpha_t \epsilon_t s R - f\left(\epsilon_t\right) - \epsilon_t c \right) \, \mathrm{d}t \right]$$

subject to $\dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right)$

and $\alpha_0, \alpha_1, \alpha_2, \ \ldots$ given,

where time is in years. I can drop constant P and rewrite the problem as

$$J = \max_{(\epsilon_t, t \geqslant 0)} \left[\int_0^1 e^{-rt - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left(\alpha_t \epsilon_t s R - f\left(\epsilon_t\right) - \epsilon_t c \right) \, \mathrm{d}t \right.$$

$$+\int_1^2 e^{-rt-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right)-\epsilon_t c\right)\,\mathrm{d}t$$
$$\left.+\int_2^3 e^{-rt-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right)-\epsilon_t c\right)\,\mathrm{d}t + \dots\right]$$
subject to $\dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right)$

and $\alpha_0, \alpha_1, \alpha_2, \dots$ given,

or

$$J = \max_{(\epsilon_t, t\geqslant 0)}\left[\sum_{i=0}^{\infty} e^{-\int_0^i \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\int_i^{i+1} e^{-rt-\int_i^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right)-\epsilon_t c\right)\,\mathrm{d}t\right]$$
subject to $\dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right)$

and $\alpha_0, \alpha_1, \alpha_2, \dots$ given.

Given that certain belief levels $\alpha_1, \alpha_2, \dots$ must be reached by the end of each year, the problem is separable for every audited reporting period. For each disclosure period that begins at time T_1 and ends at time T_2, the ex post maximization problem for the entrepreneur is

$$J = \max_{(\epsilon_t, t\in[T_1,T_2])}\left[\int_{T_1}^{T_2} e^{-rt-\int_{T_1}^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right)-\epsilon_t c\right)\,\mathrm{d}t\right]$$
subject to $\dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right)$

and $\alpha_{T_1}, \alpha_{T_2}$ given.

The solution to this problem can be found in Appendix I.A. It is a system of first order ordinal nonlinear differential equations

$$\begin{cases} \dot{\epsilon}_t = \alpha_t \dfrac{f'\left(\epsilon_t\right)\epsilon_t - f\left(\epsilon_t\right)}{f''\left(\epsilon_t\right)} - r\dfrac{\alpha_t sR - f'\left(\epsilon_t\right) - c}{f''\left(\epsilon_t\right)}, \\ \dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right) \end{cases}$$

and boundary conditions that are specified by the level of beliefs at the beginning of the disclosure period, in this case, α_{T_1}, and at the end the disclosure period, α_{T_2}. Notice that this is the same system that describes the time-paths of the experimentation rates and the posterior belief levels in the basic equilibrium case. The only difference is that for the basic equilibrium case, it is defined for the whole interval from the beginning to the end of time, with boundaries of α_0 and $\alpha_\infty = \underline{\alpha} = \frac{c}{s^{**}R}$, and for the reporting case, it is defined separately for each reporting period.

6.2. Phase One of the Game

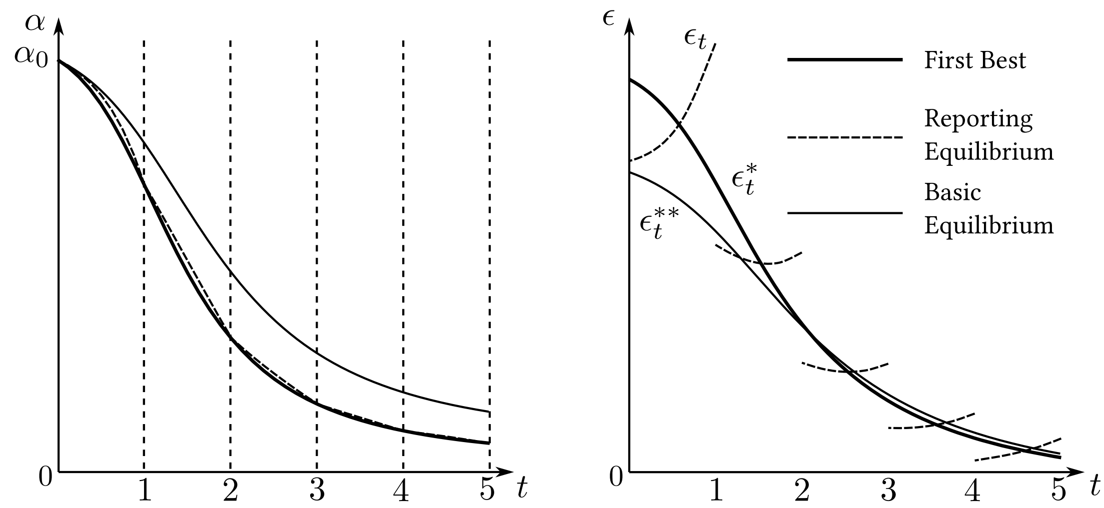
At phase one, the entrepreneur asks for share s and enough funds to finance the experiments at phase two. Everything that was derived about this phase for the basic equilibrium without audited expenditures applies in this case: the entrepreneur will want to have share s as high as possible until the budget constraint binds, then she will want to have the constraint as relaxed as possible, so that the budget constraint multiplier will be zero. Therefore, the entrepreneur solves

$$
\begin{aligned}
\max_{s} \quad & \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \left[\alpha_t \epsilon_t R - f(\epsilon_t) - \epsilon_t c\right] \mathrm{d}t \\
\text{subject to:} \quad & \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \alpha_t \epsilon_t (1-s) R \,\mathrm{d}t = \int_0^\infty e^{-tr} \epsilon_t c \,\mathrm{d}t, \\
& \dot{\alpha}_t = -\alpha_t \epsilon_t (1-\alpha_t), \\
& \alpha_0 \text{ given, } \alpha_0 \in [0,1],
\end{aligned}
$$

keeping in mind that at phase two she will have to report her aggregate expenditures every year, the reports will have to show expenditure levels equal to the aggregate efficient expenditure levels for the corresponding periods, and that her experimentation rate at phase two will follow a certain path depending on the share.

The solution involves constructing the experimentation path depending on a given share and then finding the share that satisfies the budget constraint. An example of the evolution of the posterior belief α over time and the experimentation rate produced when the entrepreneur is committed to experiment at efficient aggregate levels as measured by an auditor cumulatively at the end of each reporting period is shown in Figure I.9. It clearly shows that committing to reach certain efficiency belief levels at equally displaced intervals is efficiency-promoting as the entrepreneur who is most interested in experimenting at the efficient rate is actually required to reach certain posterior belief milestones. Notice, however, that the graphs reveal rushing nature of work under strict deadline requirements. At the beginning of the reporting period, the entrepreneur starts at a rate lower than the efficient rate. However, by the end of the reporting period she reaches the rates that are higher than the efficient rates. This is because at the beginning of the period the chances that success happens before the end of the reporting period are relatively high: the entrepreneur can slack off. By the end of the period, the entrepreneur

Figure I.9. Evolution of Beliefs and Experimentation Path Under Audited Disclosure



needs to experiment more to compensate for the lack of experimentation effort at the beginning to satisfy the reporting targets. Another reason for this is cost optimization: shifting costs to later periods reduces the total experimentation costs. However, it is clear that reporting more often is better than reporting less frequently, and that reporting does not solve the inefficiency problem completely in this crowdfunding model with convex effort costs.

Under some circumstances, optimal share in the audited reporting case can become negative. This will most likely be true for cases with low prior belief that the project is good, low surplus R, and high marginal monetary costs c. This is due to the situation in which total requested funding is higher than the expected total revenue the project generates. It will never be possible in the basic equilibrium cases because shares lower than zero mean no ex post incentives to experiment. In the audited reporting case, however, the entrepreneur will still have to experiment because she will have to report the total expenditures on the project every period. Technically, a negative share means that the investors will provide credit to the entrepreneur, and when the project succeeds they will receive the whole surplus produced by the project plus some funds they provided to the entrepreneur back. The entrepreneur will not receive any benefits from the project itself, but since she will get more funds upfront than the project is expected to ever generate, she will be better off when the project succeeds as she will still keep some of the remaining funds for herself.

The main findings of studying the audited reporting case in crowdfunding can be summed up in Proposition I.3:

Proposition I.3. *If the entrepreneur's effort costs are convex, reporting is costless, the entrepreneur is required to truthfully disclose her audited aggregate project expenditures periodically, and she is committed to efficient levels of aggregate expenditures by the end of each reporting period, then reporting is Pareto improving but still inefficient.*

Proof. The main portions of the proof are located in Appendix I.E. The intuition is simple.

The audited reporting is still inefficient because the experimentation paths produced by solving the maximization problem for the periodic audited reporting and the first best cases can only coincide if the share offered in the reporting case is equal to one. This is impossible as then the investors will not receive anything when the project succeeds and so they will not be willing to finance the project.

The audited reporting is Pareto improving when the costs of reporting are ignored because then the maximization problem the entrepreneur solves at phase one involves constraints on belief levels that coincide with the first best levels. In this sense, the constraints are more relaxed than the deterministic belief level constraints of the basic equilibrium case, and so the maximized value function is higher. □

6.3. Reporting and Audit Costs

Costless periodic reporting improves efficiency, but what if there are costs involved? It is obvious that in realistic situations there must be some positive costs associated with the audit and reporting. However, the role of the costs is simple: as long as the costs of reporting are lower than the efficiency gains from reporting, the entrepreneur is better off committing to the aggregate efficient levels of experimentation and to credible reporting of these levels. I show that the reporting costs do not affect the experimentation paths directly when the reporting periods and targets are set.

Suppose that, like in the situation described above, time is in years, audit happens at the end of each year, and the entrepreneur commits to report the aggregate experimentation levels equal to the efficient levels. Suppose that the costs associated with the preparation and audit of a single yearly report are K. Then the total costs of audited reporting are

$$e^{-r}M_1K + e^{-r2}M_2K + e^{-r3}M_3K + \ldots.$$

The costs are multiplied by $e^{-rt}M_t$, $t = 1, 2, 3, \ldots$, because it is possible that the project succeeds before the reports will have to be released.

Given that

$$M_t = \frac{1-\alpha_0}{1-\alpha_t},$$

the total audited reporting costs are equal to

$$(1-\alpha_0)\,K\left[\frac{e^{-r}}{1-\alpha_1^*} + \frac{e^{-2r}}{1-\alpha_2^*} + \frac{e^{-3r}}{1-\alpha_3^*} + \ldots\right] = (1-\alpha_0)\,K\sum_{t=1}^{\infty}\frac{e^{-rt}}{1-\alpha_t^*},$$

where α_t^* is the belief level that is reached at time t when the efficient experimentation path is followed. I established above that committing to reach certain aggregate experimentation or expenditure levels that equal to the aggregate levels of the efficient rates over the same period of time is equivalent to committing to reach the efficient levels of beliefs at the ends of the reporting periods. Therefore, the reporting costs are not affected by the experimentation rates, as α_t^* is fixed for all $t = 1, 2, 3, \ldots$. The expected audited reporting costs are treated as fixed.

It means that the entrepreneur can simply determine the gains from audited reporting ignoring the costs and then compare the gains to the costs calculated above. If it is worth it, then she should go with the audited reporting, if not—stick with the basic equilibrium. It is beyond the scope of this chapter to determine the optimal reporting schedule that allows to maximize the gains from reporting given the reporting costs, K, but this problem can be easily solved by considering different lengths of the reporting period and picking the best reporting scheme.

7. Conclusions

Crowdfunding is a new Internet-enabled way to fund startups. Recent changes in the regulatory requirements, in particular, the JOBS act, provided the legal grounds for equity crowdfunding while significantly deregulating the market and reducing the disclosure requirements. I show that the forms of crowdfunding based on the conditional rewards are inefficient and wasteful. I also demonstrate that imposing regular strict disclosure requirements can be Pareto improving. Therefore, lifting the disclosure requirements is expected to be harmful to the industry per se. However, due to possible high costs of reporting and audit, it may be beneficial to facilitate the development of specific auditing products that are affordable to project founders and satisfy the needs of crowdfunding industry.

I develop a model of interaction between the entrepreneur and investors in crowdfunding. The model includes convex effort costs and savings. I show that crowdfunding with conditional share-based rewards is inefficient. Even if the entrepreneur has full bargaining power and is willing to maximize the ex ante total surplus, the impossibility to commit to experimenting at an efficient rate ex post undermines all the efforts to achieve efficiency. The main reasons for inefficiency are liquidity constraints: the entrepreneur needs to receive money from the external sources upfront; the need to compensate the investors by offering them a share in the project surplus, which directly affects the ex post incentives; the riskiness of the project and the unobservability of the entrepreneur's actions. I show that in equilibrium experimentation rates decrease in time and that the equilibrium policy function is everywhere below the efficient policy function.

I show that deregulation of investment industry and loose reporting requirements can jeopardize the effort of the entrepreneurs to achieve efficiency. In fact, entrepreneurs may benefit from stricter regulation and reporting requirements as they will be able to use mandatory reporting as a commitment device and improve the ex post experimentation rates. This result will largely depend on the costs of reporting and auditing. If the auditors can come up with specific products tailored to the needs of the crowdfunding industry provided at reasonable price, then it will be the best outcome that does not involve regulatory intervention.

I reveal that the entrepreneur's will ask for enough funds from the investors to continue experimenting even if the project yields no success. In the event of the success, the entrepreneur will keep the remaining funds for her personal consumption. The sum the entrepreneur receives from the backers is higher than she actually expects to spend on the project, because the project might succeed at some point in time. Therefore, in equilibrium, it is expected that some of the funds will be spent on the entrepreneur's personal consumption, which is a waste from the perspectives of efficiency.

I reveal that the entrepreneur experiments with the decreasing rates over time if she keeps seeing no success. Experimentation never stops if the project keeps yielding no success, but eventually decreases to being negligible. In the equilibrium, the entrepreneur begins with lower experimentation rates than she should to achieve efficiency. It means that she learns about the project at a slower rate and stays optimistic about the project longer.

I reveal some comparative statics properties. The experimentation rates improve in the surplus size of the project and in the level of impatience, and decrease in the marginal monetary costs of experiments. It means that, as it is expected, the projects with higher expected returns or lower costs will be worked on with greater enthusiasm. The role of impatience, represented by the discount coefficient is also clear: more impatient players will not be willing to wait long until the event of success, their current experimentation rates will be higher.

There are some aspects of the model that might reveal more about crowdfunding in the future. It will be interesting to study how entrepreneurs can attract more funds by the means of marketing, and how projects are funded if the funding goals are not satisfied or surpassed. Future research directions also include models with unobserved private valuations of the parties, and analyzing optimal procurement problems with shadow costs of regulation.

Chapter II

Funding Experimentation by Angel Investors

1. Introduction

THE SUCCESSFUL FUNDING of startup firms is crucial to the dynamism and health of the economy. Angels investors play important roles in the success of startup firms by providing funding at the early stages of startup formation. Angel investors are wealthy individuals who meet the U.S. Securities and Exchange Commission definition of an accredited investor: they must have an individual net worth excluding the primary residence higher than $1 million, or they must have a stable income of more than $200,000 a year ($300,000 for married couples). They finance startups on their own or in groups. Angels provide funds to startups at early stages filling the gap between funding by friends and family and funding by venture capitalists. Employing the theory of dynamic experimentation, I recover the best contract that the entrepreneur can agree on with an angel investor when the entrepreneur has a unique project and has a power to make a take-it-or-leave-it offer to the investor. I show that the optimal contract involves funding unconditional on the project success or failure and dynamic shares. I compare experimenting paths and funding outcomes under the optimal contract with the results of signing two other contracts: the contract that involves conditional payments and the contract that is renegotiated every time period.

1.1. Stages of the Startup Development

The process of creating of a new product can be separated into three main stages shown in Table II.1. At each stage, key goals of the startup founders and the dominant funding sources are

Table II.1. Stages of the Startup Formation

Funded by	Self	Friends and Family	Individual Investors		Venture Capitalists	IPO
Startup Stage	**Formation**		**Validation**		**Scaling**	
Key Goals	• Idea of a Project • Finding Co-Founders		• Viable Product • Marketable		• Growth • Extending Markets	

different. The focus of this chapter is on the "validation" stage of the startup, in particular, on the phase where the startup is funded by the individual investors.

At the very beginning, a startup is nothing but an idea of a product, an idea which exists only in the imagination of the startup founder. As the idea matures, becomes less abstract and more concrete, the founder begins to incur costs associated with making the idea into something real. Self-funding and funding by friends and family are the main sources of capital for the fledgling startup at this stage. The uncertainty is high, and the funds are often provided just as a sign of support, without expectations of any significant return.

After experimenting with the idea for some time and working on the project seriously, the startup founder eventually reaches the stage when she needs to develop a "minimum viable product,"—a scaled-down version of the potential product that retains only the most important aspects of the original idea—to prove to herself and to the future investors that the project is potentially feasible. Usually, at this stage, it is still too early for the venture capitalists to enter since they need to be sure that the idea is definitely working before they invest. At the same time, at this point, the possibilities to raise additional capital from friends and family are typically exhausted. This is the time when angel investors come into play.

Attracted by the possibility of high gains (seeking to return about ten times the invested capital in five years), angel investors typically invest between $25,000 and $100,000 individually into the project in exchange for equity. Some angels unite in a pool to fund the project together. The risks associated with this stage of investment are high and originate in technical and marketing spheres: the product may be hard or expensive to create, or the consumers may be reluctant to buy it in large numbers. Angel investors tend to perform some background checks of the project, but the investigation is usually not thorough. In some cases, angel investors participate in the startups as board members or even employees, but usually they are only interested in the success of the business and not in the control of the firm.

The last stage of the startup formation is scaling. The product, or the prototype, exists and it is marketable. The startup begins to receive revenue, it has customers, and it is ready to expand. At this stage, venture capitalists tend to be the main source of funding for the startups, as angel investors are no longer able to satisfy the growing needs for capital. Venture capitalists invest on the magnitude of millions, investigate startup documents comprehensively, and the level of control is typically high. They can make hiring and firing decisions, and in some cases may even decide to get rid of the original founder of the startup firm. The risks are lower than at the previous stages, but still exist, as the project may or may not be scalable.

The attention of this chapter is concentrated around the "validation" stage of the startup formation and around the entrepreneur's incentives to conduct experiments. Being funded by the individual investors with scant oversight and control affects the incentives of the experimenting entrepreneurs with convex effort costs, an ability to reallocate funds in time, and an opportunity to make take-it-or-leave-it offers to the investors. The main question of this chapter is what is the best feasible contract that maximizes the incentives for the entrepreneurs to experiment. There exist several problems that must be solved to address this question.

First, there must be constructed a model of interaction between the entrepreneur and individual investors that captures the important properties of such an interaction in the economy. Second, it is important to know how the efficient interaction between the entrepreneur and investors unravels in time, and what makes it efficient. Third, it must be understood what the best contract is and how it is different from the efficiency and other possible contracts.

The solutions to these problems provide the grounds for the main results of this chapter.

1.2. Main Results

The first contribution of this chapter is in solving the dynamic experimentation model from the perspectives of time zero. I uncover the best possible contract that the entrepreneur and the angel investors can sign in the case of developed credit markets and convex experimentation costs. This contract provides a benchmark for the other possible contracts that can be signed in a more realistic environment: a contract with conditional funding, and the constantly renegotiated contract.

The second contribution is in showing that conditional payments are detrimental to the efficiency and should be avoided as they create the incentives to delay experimentation in the unobservable effort environments. The practical recommendation here is that projects should be funded upfront or unconditionally over time, such that the entrepreneur does not have additional incentives to delay experimentation. In practice, this is how angel investors actually fund the startups: they just buy shares in the project and wait until it succeeds or fails.

I demonstrate that even in the best possible scenario, the entrepreneur with convex effort costs, ability to reallocate funds in time, and full bargaining power has insufficient incentives to finance the project efficiently when funded by individual investors. Despite having full bargaining power ex ante, the entrepreneur is unable to commit to the efficient experimentation path ex post as her actions are unobservable, the project is risky, and she must share the future surplus of the project with the investors. The need to share the portion of the surplus in order to obtain funding decreases the incentives to conduct experiments efficiently ex post. The riskiness and unobservability of the actions make it impossible to solve the problem using contracts alone.

Finally, I show that if the project results in private externalities for the investor of the entrepreneur, then these externalities will be completely internalized in the contract. It means that what matters for the contract is the total social surplus generated by the project, not the private valuations of the project outcomes per se.

1.3. Related Literature

The model produced in this chapter is a model of dynamic experimentation. It describes the process of R&D as a sequence of experiments that may or may not succeed over time. The models of dynamic experimentation help capturing the dynamic nature of the moral hazard problem in the area of entrepreneurship and research. Dynamic experimentation models became popular at the beginning of the the century and are well known within the field of economic theory.

In this chapter, I construct an extension to the model by Bergemann and Hege (2005). I keep its main properties while adding more. First, instead of the linear effort costs used in the original model, I use convex effort cost. Second, I add an intertemporal budget constraint to the model as opposed to the static budget constraint of the original. Third, I allow the entrepreneur and the investors to have individual valuations of the surplus the project might generate. Finally, I solve the model fully in continuous time from the perspectives of the moment the contract is signed, not in discrete time, and not using Markov equilibrium. The main differences in the results arise from having these additional properties.

Using convex effort costs allowed me to smoothen the experimentation paths and unsure that the entrepreneur's incentive constraint always binds. In Bergemann and Hege (1998), the incentive constraint may or may not bind, creating the wide variety of paths, with kinks and jumps, depending on the parameters of the model. In my model, the entrepreneur has an ability to relocate funds over time. It means that instead of satisfying the budget constraint every instance of time, the entrepreneur just needs to satisfy the intertemporal budget constraint when the contract is signed. These properties allow the entrepreneur to write a contract that offers much more flexible incentives to experiment resulting in smooth continuous experimentation paths.

The main focus of Bergemann and Hege (1998) is on the Markov sequential equilibrium, while I concentrate on the Sequential equilibrium constructed from the perspectives of time zero. There are difficulties with long-term contracts in the environments without convex costs that stem from the fact that projects get abandoned in finite time. This is why some equilibria become tricky to sustain. In my model, the experimentation never stops until success happens, so there are no problems with commitment to the terms of the contract. I include the discussion of renegotiable contracts in the analysis to demonstrate the differences in the contractual outcomes between the optimal contract described in this chapter and the renegotiable contract.

I already discussed the important examples of the dynamic experimentation and convex costs literature in the previous chapter. In addition to these, I need to mention Heidhues et al. (2015), who model individual valuations in bandit models. The model in their paper is significantly different from my model, however, as it aims to develop a different branch of bandit literature—experimentation in teams.

1.4. Structure of the Chapter

I characterize the model of funding by angel investors. After that, I describe the first best scenario, when the social planner decides how to fund the project. Then, I explain how the equilibrium is different from the efficient solution, after which I conclude.

2. The Model

This model is a further development of the models of dynamic experimentation.

2.1. Players

Two players interact in this game: the entrepreneur and the angel investor.

The entrepreneur : the entrepreneur is risk neutral. She has a promising project and seeks funds to finance it. The entrepreneur does not have funds of her own to spend on the venture. She holds all the bargaining power to make a financing offer to the investor in this game. This is due to the assumption that there are many identical individual investors who are eager to participate in the project. The entrepreneur is solely responsible for conducting experiments after the contract with the angel investor is signed. Everything the entrepreneur will do to make the project a success will not be observed by the investor.

The angel investor : the investor is risk neutral. He is a wealthy individual and has enough funds to finance the project in exchange for the share of the returns. The investor can either accept or reject the offer proposed by the entrepreneur. Otherwise, the investor's role is rather passive. He just provides funds to the entrepreneur and hopes to receive good news regarding the project success (and his share of the surplus) some day in the future.

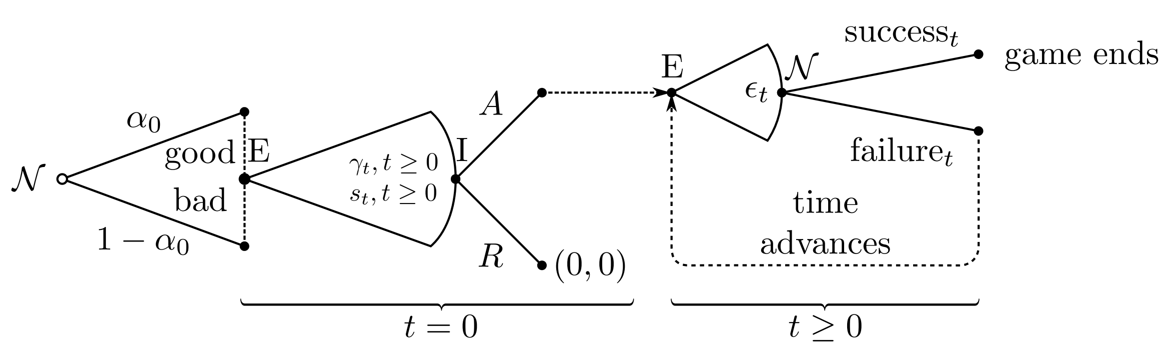
2.2. Actions

The flow of the game is presented in Figure II.1. Time is continuous and infinite. Before the game begins, Nature ($\mathcal{N}$) determines if the project is good or bad. This move is unobservable to the other players, and even the social planner cannot know if the project is good or bad. However, there is a common belief that the project must be good, α_0, the prior probability. After it has been determined by Nature if the project is good or bad, the first phase of the game begins. At phase one, the parties design and sign a funding contract.

To understand the logic of the first phase, I need to explain what happens at phase two of the game because the terms of the contract that can be designed at phase one depend on the outcome of the second phase of the game. At phase two, only the entrepreneur can take actions. Based on the terms of the contract, the prior belief that the project is good, and other parameters of the model, she produces the optimal experimentation (or effort) path $(\epsilon_t, t \geqslant 0)$. The experimentation path and its elements cannot be observed by the investor or any third party.

If the project is good, then it can succeed every period for which the experimentation rate is non-negative. This is indicated as another move by Nature in Figure II.1. This move happens every period t, right after the entrepreneur exerts effort ϵ_t, based on the intensity of it. The higher the experimentation rate, the higher the arrival rate of success. If the project succeeds, it generates a surplus. Since all the possibility to benefit from working on the project is exhausted

Figure II.1. Game Scheme



the moment it succeeds there is no need to continue experiments after that. Therefore, the game ends with the project success.

At phase one of the game, the entrepreneur (E), who has a project but does not have funds to finance it, makes a take-it-or-leave-it offer to the angel investor (I). The set of the offers the entrepreneur can propose to the investor can be rich. However, due to the unobservability of the entrepreneur's actions, there only two types of contingency possible in this model: contingency on time and contingency on the project success. Also, the entrepreneur's actions cannot be a part of the contract: the entrepreneur can pick them on a whim, without any consequences. Hence, within the game setup, the only terms that can be included in the contract are transfers that can depend on time and success of the project.

Contingency on success is simple: since success happens only once, and the game ends after the project succeeds, then it is enough to define transfers for every t conditional on success happening at t. Another component of the contract is the stream of transfers that happen before the project succeeds. Suppose that the stream of contingent transfers from the entrepreneur to the investor is a mapping

$$(t, X) \mapsto z(t, X),$$

where t is time and X is a variable that takes the value of the time the project succeeds after it happens and it is equal to ∞ before the event of success. Then for every t,

$$z(t, X) = \begin{cases} -\gamma_t c, & \text{if } X = t, \\ (1 - s_t) R, & \text{if } X < t, \\ 0, & \text{if } X > t. \end{cases}$$

The transfers look the way they do for the reasons described below. For now, they can be treated as just numbers. Without loss of generality, the contract the entrepreneur proposes contains these components:

1. A description of the project, which in this model is just α_0, the probability that the project is good.

2. The stream of payments the entrepreneur receives from the investor before the project succeeds $(\gamma_t, t \geqslant 0)$. The implicature is that the funds will be used by the entrepreneur to conduct the experiments.

3. The description of the transfer that the entrepreneur promises to the investor upon success, $(1 - s_t)\, R$, defined for every time t.

Notice that the only components that depend on time are γ_t and s_t, everything else is fixed and known. Therefore, it is enough to just describe these components in the contract. This contract is a take-it-or-leave-it offer that the entrepreneur makes to an angel investor. It the offer is rejected, the game ends with zero payoffs for both players. This is an indication of the full bargaining power of the entrepreneur in this model. It is certainly a limiting assumption, but it can be justified by assuming that the entrepreneur faces many potential angel investors. Another explanation is the fact that angel investors typically do not receive a large share in the business and usually do not participate in everyday startup operations. This is also a very pragmatic assumption. It allows me to demonstrate that even if the entrepreneur has full bargaining power, the interaction between the parties is still not efficient.

Thus the entrepreneur's pure strategies are

$$\text{Phase 1: } Z = \left(\left(\gamma_t, s_t\right), t \geqslant 0\right)$$
$$\text{Phase 2: } (Z, A) \mapsto \left(\epsilon_t, t \geqslant 0\right),$$

where $A \in \{\mathrm{a}, \mathrm{r}\}$ is the angel investor's acceptance or rejection of the offer. There is no history in phase one of the game, the entrepreneur can select any offer Z. In phase two, the history consists of the offer and of the investor's acceptance or rejection of the offer.

The angel investor's pure strategy is simple:

$$\text{Phase 1: } Z \mapsto A.$$

The history for the investors consists only of the entrepreneur's offer Z.

2.3. Information

There are two components of the information asymmetry in this game:

Information regarding the project : neither the entrepreneur, nor the investor know if the project is good or bad.

Information about the entrepreneur's actions : only the entrepreneur knows her level of experimentation and how much she spends on the project each time period t. This information is unobservable to the angel investor.

Given that the project is uncertain, both parties have beliefs that the project is good. At $t = 0$, both parties believe that the project is good with probability α_0. In the current formulation of the game, this is sufficient. However, it is convenient to define the posterior belief that the project is good as well.

The posterior belief that the project is good evolves over time according to the Bayes' Rule. Probability that no success is reached by time t is given by

$$\mathbb{P}\left(\text{no success by time } t\right) = \underbrace{1-\alpha_0}_{\text{project is bad}} + \underbrace{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}_{\substack{\text{project is good,} \\ \text{but has been failing}}} ,$$

where ϵ_t is the intensity, or rate, of experimentation at time t. Notice that if $\alpha_0 = 1$ and $\epsilon_t = \lambda$ for every t, then

$$\mathbb{P}\left(\text{no success by time } t\right) = e^{-\int_0^t \lambda \, \mathrm{d}\tau} = e^{-\lambda t} = 1 - F_{\text{exp}}\left(t\right),$$

where $F_{\text{exp}}\left(t\right)$ is the cumulative distribution function of the exponential distribution. Therefore, the distribution of the arrival time of success in this game resembles exponential distribution with varying rates of arrival and possibly positive mass at infinity (when $\alpha_0 < 1$).

Take, for example, two moments in time, t and $t' > t$. Then the probability that the project succeeds in-between times t and t' is

$$\begin{aligned}\mathbb{P}\left(\text{success between time } t \text{ and time } t'\right) &= \mathbb{P}\left(\text{no success by time } t\right) - \mathbb{P}\left(\text{no success by time } t'\right) \\ &= \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} - \alpha_0 e^{-\int_0^{t'} \epsilon_\tau \, \mathrm{d}\tau}\end{aligned}$$

$$= \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} \left(1 - e^{-\int_t^{t'} \epsilon_\tau \, \mathrm{d}\tau}\right).$$

Suppose that $\epsilon_\tau = 0$ for every $\tau \in [t, t']$. The the probability that the project succeeds between time t and time t' will be zero. Now, suppose that the entrepreneur increases experimentation rates every period and tries really hard to succeed, so hard that $\int_t^{t'} \epsilon_\tau \, \mathrm{d}\tau \to \infty$. Then the probability that success happens between t and t' approaches $\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}$. It shows that the probability to produce success at any given interval of time positively depends on the experimentation rates exerted during this time.

Equivalently (see Appendix A), the probability of reaching no success by time t can be expressed as

$$\mathbb{P}\left(\text{no success by time } t\right) = e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau},$$

where α_t is the posterior probability that the project is good at time t, conditional on no success reached by t. It is calculated according to the Bayes' Rule:

$$\alpha_t \equiv \mathbb{P}\left(\text{project is good} \mid \text{no success by time } t\right) = \frac{\overbrace{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}^{\substack{\text{project is good,} \\ \text{but has been failing}}}}{\underbrace{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}_{\text{no success by time } t}},$$

and it evolves over time according to the expression:

$$\dot{\alpha}_t \equiv \frac{\mathrm{d}\alpha_t}{\mathrm{d}t} = -\alpha_t \epsilon_t \left(1 - \alpha_t\right).$$

The entrepreneur observes her experimentation rates ϵ_t perfectly, so she knows α_t at any point in time. The angel investor does not observe ϵ_t ever. He can only expect the entrepreneur to experiment at certain rates and best respond to what they believe the experimentation rates will be when they agree or disagree to the funding offer.

2.4. Payoffs

The players' expected payoffs will depend on the offer made, Z, its acceptance or rejection in phase one, $A \in \{\mathrm{a}, \mathrm{r}\}$, and on the experimentation rates exerted by the entrepreneur in phase two of the game, $(\epsilon_t, t \geqslant 0)$. Another component of the payoff function is the players' belief that the project is good, α_0. When the offer is rejected, then both parties get nothing:

$$\pi_E(Z, \mathrm{r}, (\epsilon_t, t \geqslant 0)) = \pi_I(Z, \mathrm{r}, (\epsilon_t, t \geqslant 0)) = 0,$$

where $\pi_E(\cdot)$ is the entrepreneur's payoff function, and $\pi_I(\cdot)$ is the investor's payoff function. If the other is accepted the payoffs require more attention.

Conducting experiments at rate ϵ_t at time t will cost the entrepreneur $\epsilon_t c$ in monetary costs and $f(\epsilon_t)$ in effort costs. Parameter c is the marginal monetary cost of experimentation. Function $f(\cdot)$ is strictly convex and increasing. It indicates that the higher the experimentation rate the more taxing it is for the entrepreneur to conduct experiments. The properties of function $f(\cdot)$ are as follows:

$$\begin{aligned} f(0) &= 0, & f(x) &> 0 \text{ for } x > 0, \\ f'(0) &= 0, & f'(x) &> 0 \text{ for } x > 0, \\ f''(x) &> 0, & f'''(x) &\geqslant 0. \end{aligned}$$

The entrepreneur's total cost of conducting experiments at rate ϵ_t at time t is just a sum of the two cost components:

$$\epsilon_t c + f(\epsilon_t).$$

These costs are incurred every time period t until the projects succeeds. It is worth reiterating that the investor does not observe neither the entrepreneur's expenses, not the resulting experimentation rates.

If the experiments are conducted at a continuous positive rate and the project is good, it may eventually succeed at some random time T and produce the surplus. In this model, the surplus consists of three components:

1. R, the sharable component: this part can be shared between the parties.

2. E, the entrepreneur's individual component: this part is realized in full by the entrepreneur, it cannot be shared with the investor.

3. I, the investor's individual component of the surplus: this component is realized privately by the angel investor, it cannot be shared with the entrepreneur.

Thus when the project succeeds, the entrepreneur immediately gains E due to the project's success, the investor gets I, and the entrepreneur receives R, which she can share with the investor. The event of success is public and it is impossible to hide the fact that the project had succeeded. This interpretation allows the parties to have private valuations of the project success. If the project is bad, time $T = \infty$: the surplus will never be produced.

In practice, private valuations can be treated as externalities. It implies that the parties receive some indirect benefits from the sole fact of the project success, the benefits not associated with the sharable surplus it produces. For example, the investor may believe that the project is good for the environment, or that it helps building the community. The investor may care about these causes and value these effects at $I > 0$. When the project succeeds, the investor benefits even if she had not signed the contract with the entrepreneur. An example of the negative private valuation may be the case when the entrepreneur believes that the project will produce negative reputation effects valued at $E < 0$. These effects are associated with the fact of the project success, not with the share of the surplus the entrepreneur receives.

As per terms of the contract, the angel investor supplies the entrepreneur with the transfers, $\gamma_t c$, every period t until the project succeeds. These transfers are observable and verifiable by a third party, which means that if the investor signed the contract, he will want to adhere to its terms. Otherwise, he would risk a crippling fine, which he wants to avoid at all costs.

Future benefits and costs are discounted at rate r[1], and the rate is common for the parties. The entrepreneur has an opportunity to save and borrow funds against the future expected stream of payments at rate r, which is the same as the common discount rate. Thus the entrepreneur can freely reallocate funds between different periods of time without any utility effects. For example, the net utility effect of taking a loan of size x at time t and repaying it at time $t' > t$ will be

$$x - e^{-r(t'-t)} \frac{x}{e^{-r(t'-t)}} = x - x = 0.$$

[1]Not to be confused with the rejection of the offer indicator, r.

These are all the important components that affect the payoff functions.

The entrepreneur's expected payoff at time $t = 0$ conditional on the offer acceptance is

$$\pi_E\left(Z, \mathrm{a}, (\epsilon_t, t \geqslant 0)\right) = \mathbb{E}\left[e^{-rT}\left(s_T R + E\right) - \int_0^T e^{-rt}\left(f\left(\epsilon_t\right) + \epsilon_t c - \gamma_t c\right)\,\mathrm{d}t\right],$$

where the expectation is taken over random success time T. The first term is the expected revenue,

$$\mathbb{E}\left[e^{-rT}\left(s_T R + E\right)\right].$$

It indicates that if success happens at time T, then the entrepreneur receives private surplus E and share s_T of the sharable surplus, R, as per the contract with the investor. This future benefit is discounted at rate r. The second term represents expected costs accrued up to time T:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(f\left(\epsilon_t\right) + \epsilon_t c - \gamma_t c\right)\,\mathrm{d}t\right].$$

It includes costs of effort exerted every period, $f\left(\epsilon_t\right)$, and monetary costs of experimentation $\epsilon_t c$ minus the part that is covered by the transfers from the investor, $\gamma_t c$. The costs are discounted at rate r and accumulated up to the moment of success, T.

The angel investor's payoff is

$$\pi_I\left(Z, \mathrm{a}, (\epsilon_t, t \geqslant 0)\right) = \mathbb{E}\left[e^{-rT}\left[\left(1 - s_T\right) R + I\right] - \int_0^T e^{-rt}\gamma_t c\,\mathrm{d}t\right].$$

Similarly to the entrepreneur's payoff, there are expected benefits, which contain private surplus I and the investor's share of R, $\left(1 - s_T\right) R$, and expected costs.

Equivalently (see Appendix B), the expected payoffs of the parties at the beginning of the game, given the contract, Z, its acceptance, and experimentation path $(\epsilon_t, t \geqslant 0)$, can be expressed as

$$\begin{aligned}
\pi_E\left(Z, \mathrm{a}, (\epsilon_t, t \geqslant 0)\right) &= \int_0^\infty e^{-rt - \int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau}\left[\alpha_t \epsilon_t\left(s_t R + E\right) - f\left(\epsilon_t\right) - \epsilon_t c + \gamma_t c\right]\,\mathrm{d}t,\\
\pi_I\left(Z, \mathrm{a}, (\epsilon_t, t \geqslant 0)\right) &= \int_0^\infty e^{-rt - \int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau}\left[\alpha_t \epsilon_t\left[\left(1 - s_t\right) R + I\right] - \gamma_t c\right]\,\mathrm{d}t.
\end{aligned}$$

This form of payoffs is computationally and algebraically convenient and despite being less intuitive, it is still tractable. The expression inside the square brackets is the interim payoff. The first term inside the square brackets is multiplied by $\alpha_t \epsilon_t$, it means that there is uncertainty if the success will be reached. The other terms inside the brackets are certain within time t and reflect interim costs. The brackets are multiplied by the discount factor, e^{-rt}, and by the factor that reflects the fact that certain periods may not be reached because the project may succeed prior to that, $e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}$.

2.5. The Economics of the Budget Constraint

The whole reason why the interaction between the entrepreneur and the angel investor happens is because the entrepreneur does not have enough own funds to conduct the experiments. The investor provides the funds to the entrepreneur according to the contract each period t at rate $\gamma_t c$. The entrepreneur spends funds at rate $\epsilon_t c$. The budget constraint is

$$\mathbb{E}\left[\int_0^T e^{-rt} \gamma_t c \, \mathrm{d}t\right] \geqslant \mathbb{E}\left[\int_0^T e^{-rt} \epsilon_t c \, \mathrm{d}t\right].$$

It means that the expected sum of funds the entrepreneur receives must be higher than the expected sum she spends. The interpretation is easy, but the economics behind it require some explanation.

In the simplest case, the entrepreneur would spend each time t, before the experiment succeeded, exactly the amount she is given. In this case, $\gamma_t = \epsilon_t$ for every $t \leqslant T$, and the constraint is trivially satisfied. When the credit market is developed and competitive, it may be possible for the entrepreneur to have a less trivial budget constraint. It could be possible to reallocate funds between the time periods by borrowing and saving.

An example of how I see this is practice is the following story. Prior to signing the contract with the investor, no bank would lend money to the entrepreneur because the only thing the entrepreneur has is a project, which is too risky for the credit institutions. However, when the contract between the entrepreneur and the investor is signed, the situation changes. Given that the investor promised to fund the project at rate γ_t until the project succeeds, a competitive credit institution would now perceive the entrepreneur as creditworthy and would allow her to open a credit line to borrow and save against this stream of future payments at the market rate, r.

If at time t,

$$\epsilon_t c > \gamma_t c,$$

then the entrepreneur borrows from the bank. If

$$\gamma_t c \geqslant \epsilon_t c,$$

then she repays the loan or makes a deposit to finance the future expenses. Then the present value of the entrepreneur's future expected cashflow is

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\gamma_t c - \epsilon_t c\right)\, \mathrm{d}t\right].$$

If it is greater than zero then the credit institution believes that the entrepreneur will be able to repay the loan. If it is less than zero, then the bank will not expect to receive the loans back. Assuming the existence of the competitive credit market implies that this expression should be positive. It will act as the main budget constraint of the entrepreneur at phase one of the game.

2.6. Equilibrium Concept

The equilibrium concept is sequential equilibrium in pure strategies. The prior that the project is good, α_0 is shared by the parties along the equilibrium path and off the equilibrium path. The game can be said to be a game of imperfect information, where Nature moves first and determines if the project is good or bad. This move is unobserved by the entrepreneur and the angel investor. Nature always selects the project as "good" with probability α_0 and "bad" with probability $1-\alpha_0$.

The game is solved using backward induction. At phase two, when the offer is accepted, the entrepreneur determines optimal sequence of experimentation rates, $(\epsilon_t, t \geqslant 0)$ believing that the project is good with probability α_0. The shares and the funding schedules are fixed in the contract at this point and are treated as parameters. At phase one, the investor expects the entrepreneur to behave rationally in the future and produce a certain funding path based on the terms of the contract. The investor observes the contract and decides if he wants to accept or reject it based on his belief α_0 and his expected payoff in case of acceptance. Finally, knowing which offers the investors will accept or reject, and which experimentation path she will take in case of the acceptance, the entrepreneur tries to make the best take-it-or-leave-it offer to the investor.

3. The First Best

3.1. Problem Statement

I look at the first best scenario from the perspectives of the social planner. The fundamental uncertainty in this model is the uncertainty about the project. Nobody in this game, including the social planner, has an opportunity to know if the project is good or bad. If it was not the case and the social planner knew, somehow, the quality of the project, then the first best solution would be trivial and not very useful. If the project is known to be bad, then there is no reason to experiment; if the project is definitely good, then it must be worked on at a constant experimentation rate until it succeeds. This is not a very practical benchmark.

The utility functions are transferable in the game, thus the social planner can easily combine the payoff functions to produce the social expected payoff by adding the payoffs of the entrepreneur and the investor together:

$$\begin{aligned}\pi\left(Z, \mathrm{a}, \left(\epsilon_t, t \geqslant 0\right)\right) &= \pi_E\left(Z, \mathrm{a}, \left(\epsilon_t, t \geqslant 0\right)\right) + \pi_I\left(Z, \mathrm{a}, \left(\epsilon_t, t \geqslant 0\right)\right) \\ &= \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \left[\alpha_t \epsilon_t \left(s_t R + E\right) - f\left(\epsilon_t\right) - \epsilon_t c + \gamma_t c\right] \mathrm{d}t \\ &\quad + \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \left[\alpha_t \epsilon_t \left[\left(1 - s_t\right) R + I\right] - \gamma_t c\right] \mathrm{d}t \\ &= \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \left[\alpha_t \epsilon_t \left(R + E + I\right) - f\left(\epsilon_t\right) - \epsilon_t c\right] \mathrm{d}t.\end{aligned}$$

Notice that all the terms of the contract, Z, that is, the funding path, $(\gamma_t, t \geqslant 0)$, and the share path, $(s_t, t \geqslant 0)$, disappear from the payoff function. From the perspectives of the social planner, the terms of the contract are irrelevant. The only thing that matters for efficiency from the social planner's perspective is the experimentation path, $(\epsilon_t, t \geqslant 0)$. If the social planner can make the players follow the efficient experimentation path, then it does not matter what contract they write as long as it induces the experimentation path that maximizes the combined social payoff.

To write the social planner payoff maximization problem in the form of a dynamic control problem, I need to define one extra state variable (that will be shown to be redundant), the probability that success will happen after time t,

$$M_t = \mathbb{P}(T > t) = e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}.$$

Another state variable is the posterior probability that the project is still good at time t, α_t, defined as

$$\alpha_t = \mathbb{P}(\text{project is good} \mid T > t) = \frac{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}.$$

The trajectories of the state variables are characterized by the equations of motion:

$$\begin{aligned} \dot{\alpha}_t &\equiv \frac{\mathrm{d}\alpha_t}{\mathrm{d}t} = -\alpha_t \epsilon_t (1 - \alpha_t), \\ \dot{M}_t &\equiv \frac{\mathrm{d}M_t}{\mathrm{d}t} = -\alpha_t \epsilon_t M_t. \end{aligned}$$

Both equations of motion do not depend specifically on time t, so they are both autonomous.

I have enough information to formulate the social planner's problem in the form of the dynamic control problem:

$$\begin{aligned} &\max_{(\epsilon_t, t \geqslant 0)} \int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t (R + E + I) - f(\epsilon_t) - \epsilon_t c\right] \mathrm{d}t \\ &\text{subject to: } \dot{\alpha}_t = -\alpha_t \epsilon_t (1 - \alpha_t), \\ &\qquad \dot{M}_t = -\alpha_t \epsilon_t M_t, \\ &\qquad \alpha_0 \text{ given, } \alpha_0 \in [0, 1], \\ &\qquad M_0 = 1, \\ &\qquad \epsilon_t \geqslant 0, \forall t \geqslant 0. \end{aligned} \tag{II.1}$$

3.2. The Solution

The solution to the social planner's problem exists (see Appendix II.B.1), it is unique, and it can be expressed in terms of a policy function, $\epsilon^*(\alpha)$, which solves the first order differential equation,

$$r\left[\alpha \bar{R} - f'(\epsilon^*(\alpha)) - c\right] = \alpha \left[f'(\epsilon^*(\alpha)) \epsilon^*(\alpha) - f(\epsilon^*(\alpha)) + (1 - \alpha) f''(\epsilon^*(\alpha)) \epsilon^*(\alpha) \epsilon^{*\prime}(\alpha)\right]$$

with boundary condition

$$\epsilon^*\left(\frac{c}{\bar{R}}\right) = 0,$$

where

$$\bar{R} \equiv R + E + I.$$

The boundary condition ensures that the solution is unique, but it has an economic interpretation as well. Since α is the belief that the project is good, then $\frac{c}{R} = \underline{\alpha}$ is the lower bound on the belief level. If $\alpha \leqslant \underline{\alpha}$, then it is unreasonable to experiment. The interim payoff,

$$\alpha \epsilon \bar{R} - f(\epsilon) - \epsilon c \leqslant 0$$

for any $\alpha < \frac{c}{R}$ and positive ϵ. It means that the belief in the project is so low that it does not worth the effort. Given that the posterior belief can only decrease in time, if it does not worth it to experiment now, it will never be.

To fully characterize the efficient solution path, I need the efficient policy function, $\epsilon^* (\alpha)$; the original belief level, α_0; and the law of motion for α_t,

$$\dot{\alpha_t} = -\alpha_t \epsilon^* (\alpha_t) (1 - \alpha_t).$$

Using these three components, I can produce the efficient experimentation paths, ϵ_t^*.

3.3. Properties of the Efficient Solution

The efficient experimentation path is the benchmark experimentation path. If the entrepreneur could follow it, then the social surplus would be maximized. This is the desirable outcome from the perspectives of the social planner. As such, it is important to understand some of the main properties of the efficient experimentation path.

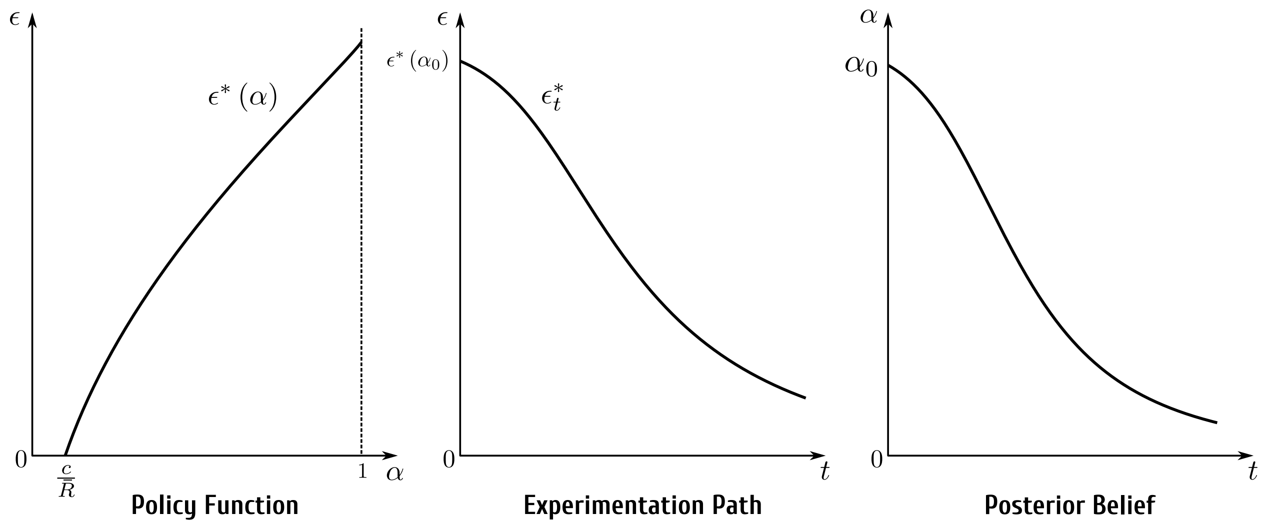
The main properties of the efficient experimentation path are summarized in Proposition II.1 and pictured in Figure II.2.

Proposition II.1. *The efficient experimentation path in dynamic experimentation models with convex experimentation costs and funding by angel investors satisfies:*

Some projects are not worth the risk *: if $\alpha_0 \leqslant \frac{c}{R}$, then the project is better left alone.*

Staticity at the top *: the experimentation rate for sure projects ($\alpha = 1$) is stationary.*

Figure II.2. The Efficient Solution



Experimentation rate strictly decreases in time *: the efficient policy function strictly increases in* α*.*

Experimenting never stops *: experiments only stop if success happens, otherwise, experimentation rate is always positive.*

Proof. The proof can be found in Appendix II.B.2. I provide intuition for it here:

Some projects are not worth the risk : some projects are so bad that it is unreasonable to spend any effort on them. Suppose that a project with $\alpha_0 \leqslant \frac{c}{\bar{R}}$ were worked on with positive experimentation rate. If the prior that the project is good, α_0, is lower than $\frac{c}{\bar{R}}$, then the posterior α_t will only decrease over time, it never goes up. Thus for every time period t, if $\epsilon_t \geqslant 0$, the interim payoff from experimenting,

$$\alpha_t \epsilon_t \bar{R} - f(\epsilon_t) - \epsilon_t c \leqslant \frac{c}{\bar{R}} \epsilon_t \bar{R} - f(\epsilon_t) - \epsilon_t c = -f(\epsilon_t) \leqslant 0.$$

Then the total social payoff will only be negative as

$$\int_0^\infty e^{-tr - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left[\alpha_t \epsilon_t \bar{R} - f(\epsilon_t) - \epsilon_t c\right] \mathrm{d}t \leqslant 0,$$

so there is no point in trying to make the project with $\alpha_0 \leqslant \frac{c}{R}$ a success. Therefore, it must be the case that for $\alpha \leqslant \frac{c}{R}$, $\epsilon^* (\alpha) = 0$.

Staticity at the top : for sure projects, with prior $\alpha_0 = 1 > \frac{c}{R}$, posterior α_t stays fixed no matter what: $\alpha_t = \alpha_0 = 1$: if you are certain that the project is good, then no matter how many times the project fails you will still remain certain that it is good. The efficient experimentation rate must satisfy

$$r \left[\bar{R} - f' (\epsilon^* (1)) - c \right] = f' (\epsilon^* (1)) \, \epsilon^* (1) - f (\epsilon^* (1)) \, .$$

This is no longer a differential equation, but just a simple ordinary equation that can be solved for $\epsilon^* (1)$, which is just a number. It stays strictly positive, finite, and constant for every time period. Therefore, for sure projects the experimentation rate is static.

Experimentation rate strictly decreases in time : for $\alpha \in \left(\frac{c}{R}, 1 \right)$, experimentation rate *increases* in α. Posterior belief α_t strictly decreases over time given no success, so the efficient experimentation rate, $\epsilon^* (\alpha)$, decreases in time.

The proof is by contradiction:

- the efficient experimentation rate is continuously differentiable;
- it is always the case that $\epsilon^* \left(\frac{c}{R} \right) = 0$ and $\epsilon^* (1) > 1$, provided that $\frac{c}{R} < 1$;
- so function $\epsilon^* (\alpha)$ must increase somewhere on $\left(\frac{c}{R}, 1 \right)$, refer to Figure II.2;
- suppose it also decreases on $\left(\frac{c}{R}, 1 \right)$, then there must be at least one local maximum in this interval;
- however, all the extreme points that can exist on $\left(\frac{c}{R}, 1 \right)$ are local minima;
- this is a contradiction to function $\epsilon^* (\alpha)$ decreasing on the given interval;
- therefore, the efficient policy function strictly increases in α on $\left(\frac{c}{R}, 1 \right)$.

The efficient experimentation rate increases in the posterior belief level, α, and, consequently, decreases in time.

Experimenting never stops : experiments never stop. For $\alpha \in \left[\frac{c}{R}, 1\right]$, the efficient policy function, $\epsilon^*(\alpha)$, is strictly increasing and is everywhere below critical function $\bar{\epsilon}(\alpha)$ from Appendix C. The critical function is constructed in such a fashion to ensure that if it is followed, then the experiments stop in finite time. Any function that is strictly below the critical function, but strictly positive on $\alpha \in \left(\frac{c}{R}, 1\right]$ does not provide enough experimentation effort to stop in finite time. Therefore, as the efficient policy function is everywhere below the critical function, efficient experimentation should not stop until the project succeeds.

$\square$

4. The Equilibrium

The social planner's solution establishes the ideal experimentation path for this game, the path that allows to produce the highest social surplus. The actual equilibrium of the game will not coincide with the efficient solution. The main reason for the inefficiency of the equilibrium is the discrepancy between the entrepreneur's ex ante and ex post incentives to conduct the experiments. Prior to signing the contract the entrepreneur wants to maximize the social surplus because she has full bargaining power in this game, but after signing the contract her incentives will be affected by the need to share the surplus with the investor and by the stream of conditional payments that she can loose if she succeeds too early.

I solve the game by backward induction. Players are sequentially rational. Thus the investor at phase one has an idea of how the entrepreneur will work on the project given the offer she proposes. Based on these expectations, the investor can accept or reject the offer. Then the entrepreneur, knowing which offers will be accepted and which rejected, can find the best contract to offer to the investor. I begin solving the model form phase two, the experimentation phase. Then I continue to phase one and discover the optimal offer that the entrepreneur is going to make to the investor.

4.1. Phase Two of the Game

The second phase of the game is the phase at which the entrepreneur conducts experiments after signing the contract. The terms of the contract are set, the stream of payments conditional on no success, $(\gamma_t, t \geqslant 0)$, and the evolution of shares, $(s_t, t \geqslant 0)$, are defined and are treated as fixed.

The entrepreneur can reallocate funds over time by borrowing and saving at rate r, which is equal to the discount rate. The main budget constraint at phase two is

$$\mathbb{E}\left[\int_0^T e^{-rt}\epsilon_t c \,\mathrm{d}t\right] \leqslant \mathbb{E}\left[\int_0^T e^{-rt}\gamma_t c \,\mathrm{d}t\right],$$

or (see Appendices A and B),

$$\int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau}\epsilon_t c \,\mathrm{d}t \leqslant \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \gamma_\tau \,\mathrm{d}\tau}\gamma_t c \,\mathrm{d}t,$$
$$\int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau}\left(\epsilon_t c - \gamma_t c\right)\,\mathrm{d}t \leqslant 0.$$

It reflects the fact that the entrepreneur cannot expect to spend more on experiments than she ever expects to receive from the investor. I will guess and verify in the description of phase one that the budget constraint will be satisfied with equality, but the multiplier associated with it will be zero. The intuition is simple: if the multiplier is positive there is an ex ante value in relaxing the constraint, so there is an opportunity to change the contract and ask for more funds.

The entrepreneur's problem at phase two is the maximization problem, which can be described from the perspectives of the dynamic programming as

$$\begin{aligned}
&\max_{(\epsilon_t, t \geqslant 0)} \left[\int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t \left(s_t R + E\right) - f\left(\epsilon_t\right) - \epsilon_t c + \gamma_t c\right]\,\mathrm{d}t\right] \\
\text{subject to: } &\int_0^\infty e^{-rt} M_t \left(\epsilon_t c - \gamma_t c\right)\,\mathrm{d}t \leqslant 0, \\
&\dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right), \\
&\dot{M}_t = -\alpha_t \epsilon_t M_t, \\
&\left(s_t, t \geqslant 0\right), \left(\gamma_t, t \geqslant 0\right) \text{ given}, \\
&\alpha_0 \text{ given}, \alpha_0 \in [0,1], \\
&M_0 = 1,
\end{aligned}$$

$$\epsilon_t \geqslant 0, \forall t \geqslant 0,$$

where $M_t \equiv e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}$. Appendix II.C.1 demonstrates that the problem with the binding budget constraint is equivalent to the problem without the budget constraint, but with the higher marginal monetary cost of experimentation, c. Therefore, to derive the solution, I can ignore the budget constraint, and rewrite the problem as:

$$\max_{(\epsilon_t, t \geqslant 0)} \left[\int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t \left(s_t R + E \right) - f\left(\epsilon_t\right) - \epsilon_t c + \gamma_t c \right] \, \mathrm{d}t \right]$$

$$\begin{aligned} \text{subject to: } & \dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right), \\ & \dot{M}_t = -\alpha_t \epsilon_t M_t, \\ & \left(s_t, t \geqslant 0\right), \left(\gamma_t, t \geqslant 0\right) \text{ given}, \\ & \alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\ & M_0 = 1, \\ & \epsilon_t \geqslant 0, \forall t \geqslant 0. \end{aligned}$$

This problem is solved in Appendix II.A. The solution is unique when functions s_t and γ_t are continuously differentiable and the limit,

$$\lim_{t \to \infty} \alpha_t = \frac{c}{\underline{s} R + E},$$

where

$$\underline{s} = \lim_{t \to \infty} s_t,$$

exists. The equilibrium experimentation path at phase two is obtained from the first order differential equation,

$$r \left[\alpha_t \left(s_t R + E \right) - f'\left(\epsilon_t\right) - c \right] = \alpha_t \left[f'\left(\epsilon_t\right) \epsilon_t - f\left(\epsilon_t\right) + \gamma_t c \right] + \alpha_t \dot{s}_t R - f''\left(\epsilon_t\right) \dot{\epsilon}_t, \tag{II.2}$$

the law of motion for the posterior belief,

$$\dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right),$$

and two boundary conditions:

$$\alpha_0 \text{ is given and}$$
$$\alpha_\infty = \frac{c}{\underline{s}R + E}.$$

An important observation regarding (II.2) is that it is defined by two time paths, $(s_t, t \geqslant 0)$ and $(\gamma_t, t \geqslant 0)$. Therefore, the same equilibrium experimentation path can be induced by different combinations of these paths. It means that there exist some equivalence between various contracts from the perspectives of the second phase of the game. At phase two, the entrepreneur will be indifferent between these contracts.

Rearrange the terms in (II.2):

$$r\left[\alpha_t E - f'\left(\epsilon_t\right) - c\right] = \alpha_t\left[f'\left(\epsilon_t\right)\epsilon_t - f\left(\epsilon_t\right)\right] + \alpha_t R\left(\dot{s}_t - rs_t\right) + \alpha_t \gamma_t c - f''\left(\epsilon_t\right)\dot{\epsilon}_t.$$

Define a virtual parameter,

$$\omega_t \equiv R\left(\dot{s}_t - rs_t\right) + \gamma_t c,$$

and differentiate it with respect to time:

$$\dot{\omega}_t = R\left(\ddot{s}_t - r\dot{s}_t\right) + \dot{\gamma}_t c.$$

If for two different contracts ω_t is the same for every period and shares, s_t, converge to the same limit, $\underline{s}$, then these two contracts are equivalent from the perspectives of the second phase of the game.

Suppose, for example, that there is a contract with fixed shares and varying payments. There exists a contract with fixed payments and varying shares that results in the same experimentation path. Suppose that shares are fixed in the contract for every time period at the limit level, $s_t = \underline{s}$. Then

$$\omega_t = -r\underline{s}R + \gamma_t c$$

and

$$\dot{\omega}_t = \dot{\gamma}_t c.$$

If I wanted to replicate the experimentation path induced by this contract using a contract with varying shares and fixed payments, I would set

$$R\left(\ddot{s}_t - r\dot{s}_t\right) = \dot{\gamma}_t c$$

and would make sure that $s_t \to \underline{s}$ as time goes to infinity. This way, I would have two contracts, one, with fixed shares and varying payments, and another, with fixed payments and varying shares, that will provide equivalent incentives to the entrepreneur and will produce the same experimentation path. There are other contracts that induce the same experimentation path as in this example. Having such a flexibility in writing contracts is a convenient feature of this model.

4.2. Phase One of the Game

At phase one, both players expect that the entrepreneur will behave according to the differential equation described above. I derive the solution for this phase of the game assuming sequential rationality. It means that along the equilibrium experimentation path, the entrepreneur behaves exactly as she is expected to behave and there is no profitable deviation from the equilibrium.

The decision to accept or reject the contract by the investor is based on the terms of the contract and the expectations of the investor regarding the funding path. Based on the common belief that the project is good, α_0, the terms of the contract,

$$Z = \left(\left(\gamma_t, s_t\right), t \geqslant 0\right),$$

and the experimentation path induced by this contract, $(\hat{\epsilon}_t, t \geqslant 0)$, the investor will sign the contract if his participation constraint is satisfied:

$$\int_0^\infty e^{-rt-\int_0^t \hat{\alpha}_\tau \hat{\epsilon}_\tau \,\mathrm{d}\tau} \left[\hat{\alpha}_t \hat{\epsilon}_t \left[\left(1 - s_t\right) R + I\right] - \gamma_t c\right] \,\mathrm{d}t \geqslant 0.$$

The entrepreneur has all the bargaining power in this game, thus it is obvious that she will offer the investor a contract with such terms that

$$\int_0^\infty e^{-rt-\int_0^t \hat{\alpha}_\tau \hat{\epsilon}_\tau \,\mathrm{d}\tau} \left[\hat{\alpha}_t \hat{\epsilon}_t \left[\left(1 - s_t\right) R + I\right] - \gamma_t c\right] \,\mathrm{d}t = 0.$$

The angel investor's budget constraint binds.

What is left to do is to find the optimal contract the entrepreneur offers to the investor. From the perspectives of phase one of the game, the entrepreneur's problem is

$$\begin{aligned}
&\max_{((s_t,\gamma_t),t\geqslant 0)} \left[\int_0^{\infty} e^{-tr} M_t \left[\hat{\alpha}_t \hat{\epsilon}_t \left(s_t R + E\right) - f\left(\hat{\epsilon}_t\right) - \hat{\epsilon}_t c + \gamma_t c\right] \,\mathrm{d}t\right] \\
&\text{subject to: } \int_0^{\infty} e^{-rt} M_t \left[\hat{\alpha}_t \hat{\epsilon}_t \left[\left(1 - s_t\right) R + I\right] - \gamma_t c\right] \,\mathrm{d}t = 0, \\
&\qquad \int_0^{\infty} e^{-rt} M_t \left[\gamma_t c - \hat{\epsilon}_t c\right] \,\mathrm{d}t = 0, \\
&\qquad \dot{\hat{\alpha}}_t = -\hat{\alpha}_t \hat{\epsilon}_t \left(1 - \hat{\alpha}_t\right), \\
&\qquad \dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t, \\
&\qquad \alpha_0 \text{ given}, \alpha_0 \in \left[0, 1\right], \\
&\qquad M_0 = 1.
\end{aligned}$$

Keep in mind that $\hat{\epsilon}_t$ is the induced experimentation path that depends on the terms of the contract.

From the first constraint, express

$$\int_0^{\infty} e^{-rt} M_t \gamma_t c \,\mathrm{d}t = \int_0^{\infty} e^{-rt} M_t \hat{\alpha}_t \hat{\epsilon}_t \left[\left(1 - s_t\right) R + I\right] \,\mathrm{d}t,$$

and combine this result with the objective function:

$$\begin{aligned}
&\int_0^{\infty} e^{-tr} M_t \left[\hat{\alpha}_t \hat{\epsilon}_t \left(s_t R + E\right) - f\left(\hat{\epsilon}_t\right) - \hat{\epsilon}_t c + \hat{\alpha}_t \hat{\epsilon}_t \left[\left(1 - s_t\right) R + I\right]\right] \,\mathrm{d}t \\
&\qquad = \int_0^{\infty} e^{-tr} M_t \left[\hat{\alpha}_t \hat{\epsilon}_t \left(R + E + I\right) - f\left(\hat{\epsilon}_t\right) - \hat{\epsilon}_t c\right] \,\mathrm{d}t.
\end{aligned}$$

Thus when the first constraint binds, the objective function does not depend explicitly on the terms of the contract. In fact, this objective function is exactly the same as the objective function in the social planner's problem, (II.1). It means that the entrepreneur captures all the surplus in this game. Her ex ante incentives are to maximize the social surplus.

Unfortunately, the entrepreneur's problem at phase one of the game is not equivalent to the social planner's problem. The entrepreneur needs to satisfy two constraints. The first constraint is the investor's participation constraint. I already argued that it will have to be satisfied with equality because then the entrepreneur captures the whole surplus in this game. The second constraint is the budget constraint,

$$\int_0^\infty e^{-rt} M_t \left[\gamma_t c - \hat{\epsilon}_t c\right] \, \mathrm{d}t = 0.$$

I argue that this constraint binds (see Appendix II.C.2). The intuition is simple. If the budget constraint does not bind, it is possible to ask for less funds and lower share and keep the induced experimentation rates intact. This way, the objective function will no be affected. However, lower funding rates and lower shares will mean that the investor will be better off. The investor's budget constraint will no longer be binding. With both constraints not binding, the problem will be similar to the social planner's problem and the first best result will have to be possible. In other words, when the budget constraint does no bind, it is possible to write a better contract. Therefore, the budget constraint must be binding.

Suppose, however, that the budget constraint is binding but restricting at the second phase of the game. As it was shown in Appendix II.C.1, in this case, there will be pure efficiency loss equivalent to an increase in the marginal monetary cost of experimentation, c. Any shadow costs incurred at the second phase of the game imply that the contract is specified in such a fashion that a part of the generated surplus is lost. It is possible to write a better contract to avoid the shadow costs at phase two (see the second argument in Appendix II.C.2). Therefore, the budget constraint will not be restrictive at the second stage.

Now, that I have established that both constraints bind at phase one of the game, I can assign multiplier λ to the investor's participation constraint and multiplier μ to the budget constraint to write the problem. Appendix II.C.3 is devoted to formulating the entrepreneur's problem with both constraints binding. The entrepreneur knows that at phase two she will experiment according to (II.2) given two paths, $(s_t, t \geqslant 0)$ and $(\gamma_t, t \geqslant 0)$ described in the contract written at phase one. Thus, at phase one she needs to solve (II.2) for ϵ_t and express it as a function of the shares and funding paths and then use the result as a constraint in the phase one problem. Alternatively, she can solve (II.2) for the share and control the induced experimentation path, $(\hat{\epsilon}_t, t \geqslant 0)$, directly together with $(\gamma_t, t \geqslant 0)$. Thus to satisfy the second phase incentives and induce experimentation rates $(\hat{\epsilon}_t, t \geqslant 0)$, shares and funding rates at every time t must satisfy

$$s_t R + E = \frac{f'(\hat{\epsilon}_t)}{\hat{\alpha}_t} + \int_t^\infty e^{-r(\tau-t)} \left[\frac{f'(\hat{\epsilon}_\tau)\hat{\epsilon}_\tau + rc}{\hat{\alpha}_\tau} - f(\hat{\epsilon}_\tau) + \gamma_\tau c \right] \mathrm{d}\tau.$$

Using this expression in the problem statement directly, I can reformulate the entrepreneur's problem at phase one as a dynamic control problem:

$$\begin{aligned} \max_{(\hat{\epsilon}_t, t\geqslant 0), (\gamma_t, t\geqslant 0), \lambda, \mu} \int_0^\infty e^{-tr} \Bigg[& M_t \left(\lambda \hat{\alpha}_t \hat{\epsilon}_t (R + E + I) + (1-\lambda) f'(\hat{\epsilon}_t) \hat{\epsilon}_t - f(\hat{\epsilon}_t) \right. \\ & \left. + (1 - \lambda + \mu) \gamma_t c - (1+\mu) \hat{\epsilon}_t c \right) \\ & + (1-\lambda)(1-M_t) \left(\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t) + \gamma_t c \right) \Bigg] \mathrm{d}t \end{aligned}$$

$$\begin{aligned} \text{subject to: } & \dot{\hat{\alpha}}_t = -\hat{\alpha}_t \hat{\epsilon}_t (1 - \hat{\alpha}_t), \\ & \dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t, \\ & \alpha_0 \text{ given, } \alpha_0 \in [0, 1], \\ & M_0 = 1. \end{aligned}$$

I characterize the best solution and then compare the experimentation paths produced by the best contract to the paths produced by two more restricting contracts: the contract that can be written in the absence of the credit market and the dynamically renegotiable contract.

4.3. The Best Contract

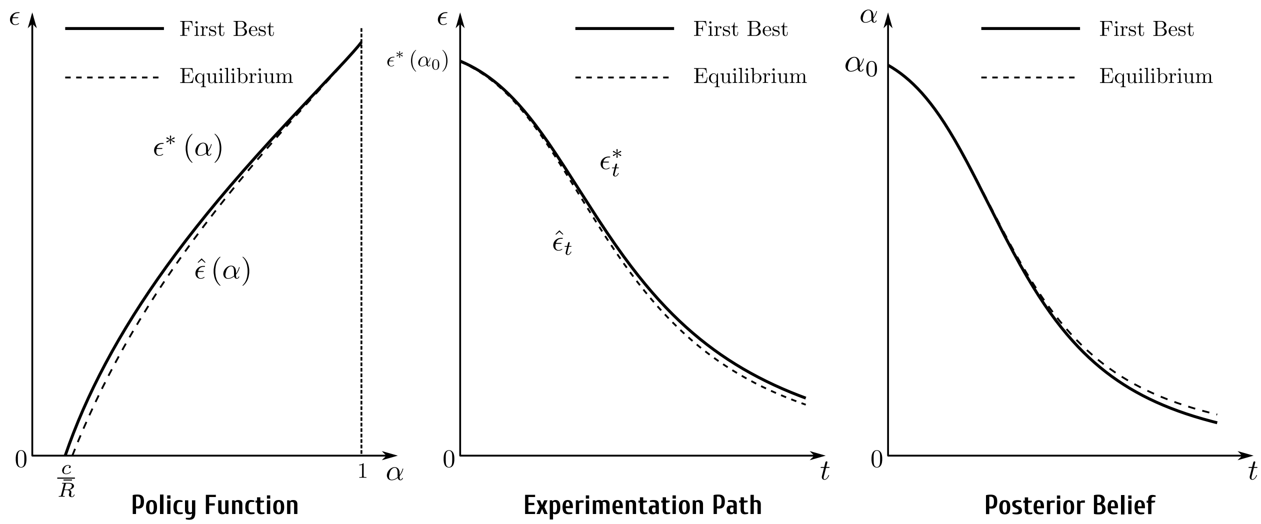
I solve the entrepreneur's problem in Appendix II.C.4. The solution consists of a first order non-linear differential equation, (II.e), and one boundary condition, thus it is unique. I do not reproduce the solution here as the equations are rather long, however, I characterize the important properties of the induced experimentation path and describe the main features of the best contract. The main results are summed up in Proposition II.2 and shown in Figure II.3.

Proposition II.2. *The best contract that the entrepreneur can propose to an angel investor in a dynamic experimentation model with convex effort costs and developed credit markets satisfies:*

Funds are provided unconditionally *: funding of the project should not stop if success happens.*

Experimentation rates are inefficient *: even the best contract does not provide enough incentives to experiment at the efficient rate.*

Figure II.3. **The Best Contract: Induced Experimentation Rates**



Externalities that the project produces for the parties will be internalized *: the experimentation rates depend on the total size of the surplus the project generates upon success, not just on the entrepreneur's share of it.*

Some projects are not worth the risk *: if* $\alpha_0 \leqslant \frac{2c}{\bar{R}}$*, then the project will not be undertaken.*

Experimenting never stops *: experiments only stop if success happens, otherwise, experimentation rate is always positive.*

Proof. The proof is located in Appendix II.C.5. I provide the intuition for it here:

Funds are provided unconditionally : This is the direct result of solving the maximization problem at phase one of the game. The economic intuition is that funds that are provided over time until the project succeeds create the unwanted incentives for the entrepreneur to delay the experimentation in order to receive the funds in the future. It is best to avoid this practice and provide the funds upfront. Unfortunately, it requires the existence of the developed credit market. However, given that when success happens the entrepreneur will receive a significant benefit, while the money provided to conduct experiments are relatively insignificant, the negative effect of having to rely on conditional funding is expected to be low.

Experimentation rates are inefficient : The inefficiency results from the need to satisfy the budget constraint. Only if the budget constraint multiplier is zero, the experimentation path produced by the best contract is the same as for efficient experimentation path.

Externalities that the project produces for the parties will be internalized : The private externalities which the parties receive when the project succeeds, E and I, only matter as the parts of the total surplus, $\bar{R} = R + E + I$. As long as the total surplus is the same, the distribution of the private and the sharable valuations of the project does not matter. This is an example of how the externalities of the participants are internalized by the means of the contract.

Some projects are not worth the risk : If the prior belief that the project is good is lower than $\frac{2c}{\bar{R}}$, then the posterior beliefs will only be lower than that in the future. This belief level is insufficient to satisfy the participation constraints of the players and it is not enough to produce a single experimentation session with the positive experimentation rates, let alone carry on with experiments for some prolonged time. Therefore, some projects that should be worked on from the perspectives of the efficiency will be left out if $\alpha_0 \in \left(\frac{c}{\bar{R}}, \frac{2c}{\bar{R}}\right]$.

Experimenting never stops : For the experiments to only stop when the project succeeds and otherwise continue indefinitely, the policy function must not increase abruptly at the point of the lowest belief level. Following the logic described for the first best policy function, I show that this is, indeed, the case for the policy function produced by the best contract. Therefore, the experiments financed by the angel investors will be carried out until the success happens or forever. The best contract has no expiration date.

$\square$

4.4. Alternative Contracts

In order to understand the difference between the best contract and the other possible funding schemes, I compare the outcomes of funding the project under the terms of the best contract with the outcomes of funding the project by the means of two alternative contracts that the entrepreneur can sign with the angel investor. The first alternative contract involves conditional payments. It is intended to serve as an example of the contract that can be written in the absence of the developed credit markets or in the case when the entrepreneur failed to secure a credit line and is unable to reallocate the funds over time. The second alternative contract is a renegotiable contract. It applies to the cases when the entrepreneur has to work with the angel investors who want to be able to quit financing the project at any moment in time and resume funding it some time in the future. Technically, such a contract is equivalent to having no long term contract at all and figuring out how to finance the project as the experiments go.

Consider the first alternative contract with the conditional payments. Suppose that there is no way to reallocate the funds between the time periods. Then it must be the case that, for every t,

$$\gamma_t = \hat{\epsilon}_t,$$

and so the budget constraint is satisfied automatically. Everything else is identical to the general problem of finding the equilibrium. The investor's participation constraint must still bind,

$$\int_0^{\infty} e^{-rt} M_t \left[\hat{\alpha}_t \hat{\epsilon}_t \left[(1 - s_t) R + I\right] - \hat{\epsilon}_t c\right] \mathrm{d}t = 0.$$

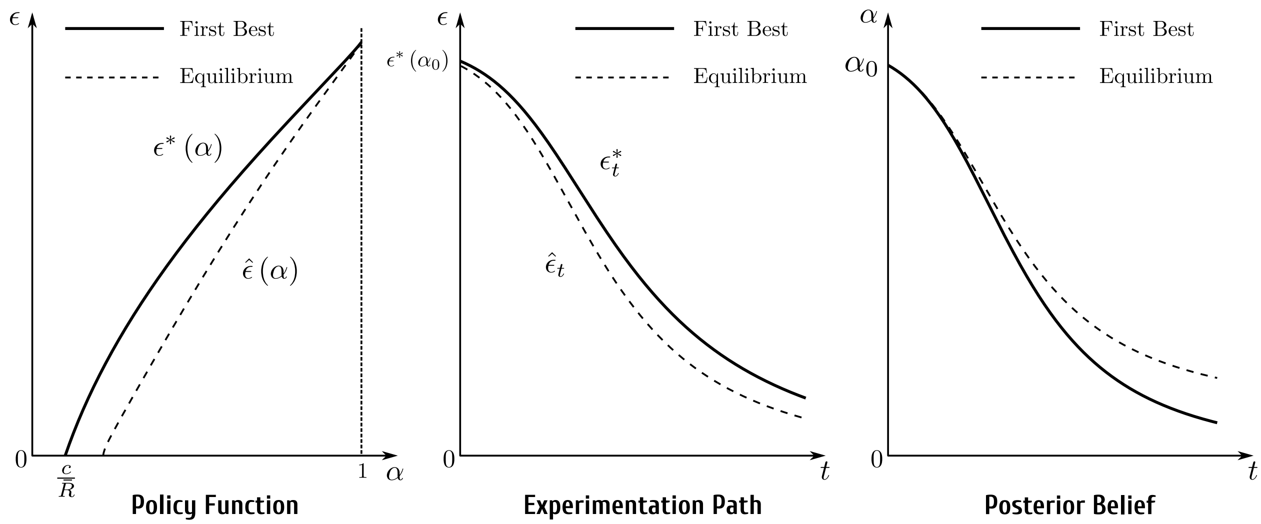
I follow the logic of Appendix II.C.1, while keeping all $\gamma_t = \hat{\epsilon}_t$ and write the maximization problem that the entrepreneur faces at the first phase of the game:

$$\begin{aligned}
\max_{(\hat{\epsilon}_t, t \geqslant 0), \lambda} \int_0^{\infty} e^{-tr} \Bigg[& M_t \left(\lambda \hat{\alpha}_t \hat{\epsilon}_t \left(R + E + I\right) + (1 - \lambda) f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right) - \lambda \hat{\epsilon}_t c\right) \\
& + (1 - \lambda)(1 - M_t) \left(\frac{f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f\left(\hat{\epsilon}_t\right) + \hat{\epsilon}_t c \right) \Bigg] \mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\text{subject to: } & \dot{\hat{\alpha}}_t = -\hat{\alpha}_t \hat{\epsilon}_t \left(1 - \hat{\alpha}_t\right), \\
& \dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\
& M_0 = 1.
\end{aligned}$$

I solve this problem in Appendix II.C.6. The resulting differential equation and the boundary condition that I produce look very similar to the equilibrium conditions derived for the best

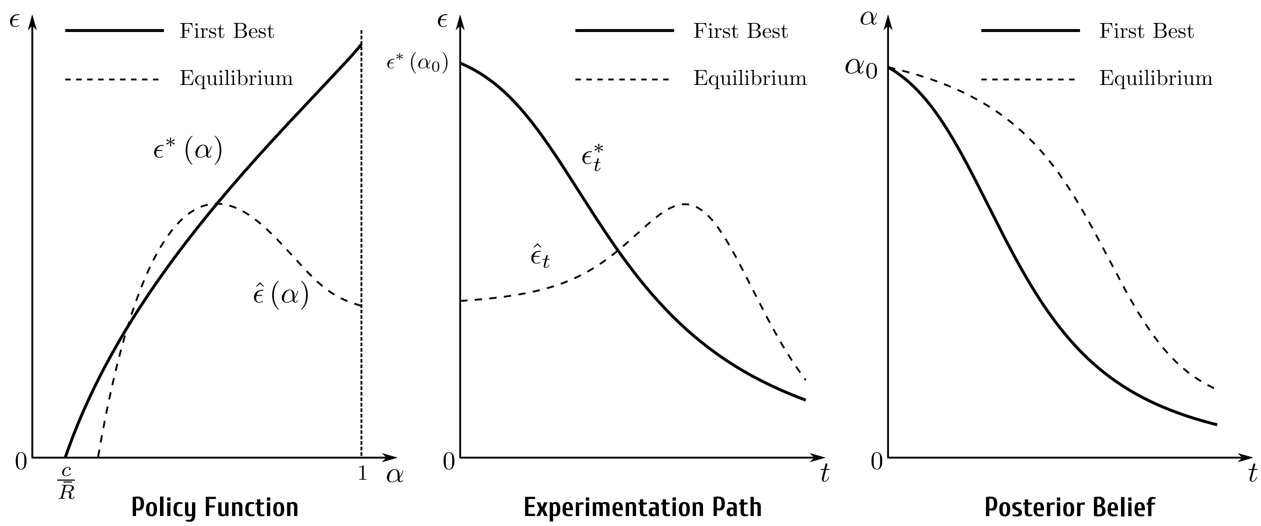
Figure II.4. The Contract with Conditional Funding



contract. Due to the complexity of these expressions, they are left in the Appendix. Figure II.4 shows the policy function, the experimentation path, and the belief evolution path produced by the entrepreneur as the result of signing and following the contract with the payments conditional on no success. The paths and the policy function depicted in the figure look somewhat similar to the paths and the functions shown in Figure II.3, which demonstrates the experimentation rates and beliefs produced by the best contract. However, the policy function is lower and the belief evolution happens at a slower rate, so I can tell that the contract with the conditional funding will be worse for the entrepreneur. Moreover, the best contract is not constrained by the requirement that $\gamma_t = \hat{\epsilon}_t$ for every time t. Thus the entrepreneur can achieve more.

Now, consider the second alternative contract that the entrepreneur can write and sign with the angel investor. This contract involves renegotiation of the contract terms that happens every time period. The reasons why such a contract will be signed may be different: it may be because the investor is unable to commit to the terms of a complicated contract or because the entrepreneur wants to switch between different investors from time to time and thus needs the contract with the possibility to exit without penalties. In any case, it is an interesting alternative to consider. In a sense, this contract will be as good as having no long term contract at all.

Figure II.5. The Renegotiable Contract



Given that the contract can be renegotiated at will, the entrepreneur will have to ask for conditional funds:

$$\gamma_t = \hat{\epsilon}_t.$$

She will also have to satisfy the immediate participation constraint of the investor:

$$\hat{\alpha}_t \hat{\epsilon}_t \left[(1 - s_t) R + I \right] - \gamma_t c = 0,$$

for every time t. As the result, the budget constraint and the investor's participation constraint will be satisfied automatically. It is possible to derive the expression for the optimal share directly from the investor's participation constraint for any time t:

$$s_t = \frac{R + I}{R} - \frac{\gamma_t c}{\hat{\alpha}_t \hat{\epsilon}_t R},$$

or given that $\gamma_t = \hat{\epsilon}_t$,

$$s_t = \frac{\hat{\alpha}_t (R + I) - c}{\hat{\alpha}_t R}.$$

Therefore, all the terms of the contract are known. What is left to do is to find $\dot{s}_t$:

$$\dot{s}_t = \frac{\dot{\hat{\alpha}}_t (R + I)}{\hat{\alpha}_t R} - \dot{\hat{\alpha}}_t \frac{\hat{\alpha}_t (R + I) - c}{\hat{\alpha}_t^2 R} = \dot{\hat{\alpha}}_t \frac{c}{\hat{\alpha}_t^2 R} = -\hat{\epsilon}_t (1 - \hat{\alpha}_t) \frac{c}{\hat{\alpha}_t R}.$$

The entrepreneur's problem at stage one is solved. I can insert the expressions for s_t, $\dot{s}_t$, and γ_t into the differential equation that characterizes the equilibrium experimentation path produced at the second phase of the game, (II.2), directly and produce the condition that describes the experimentation path for the renegotiable contract:

$$r\left[\hat{\alpha}_t\left(R+E+I\right)-f'\left(\hat{\epsilon}_t\right)-2c\right]=\hat{\alpha}_t\left[f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-f\left(\hat{\epsilon}_t\right)\right]-\left(1-2\hat{\alpha}_t\right)\hat{\epsilon}_t c-f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t.$$

The boundary condition is straightforward: given that it was established that even for the best contract, no project with the prior, α_0, less that $\frac{2c}{R}$ will be worked on, it is obvious that the renegotiation can continue until the lower bound on the belief level is reached. Therefore,

$$\hat{\epsilon}\left(\frac{2c}{R}\right)=0$$

is the boundary condition for the renegotiable contract.

It is obvious that following the terms of the renegotiable contract will lead to the outcomes worse than for the best contract and for the contract with conditional payments. The reason for this is that the problem of writing the renegotiable contract is even more constrained than the problem of writing the contract with the conditional funding. However, the experimentation paths and the policy functions produced by the renegotiable contract are very different from all the paths I demonstrated before. Figure II.5 depicts the policy function, the experimentation path, and the belief evolution path produced by such a contract.

It is interesting to note that for some belief levels the policy function lies even higher than the efficient policy function, it means that the entrepreneur will experiment at a higher rate than it is needed. This possibility of overexperimentation is a direct consequence of having the convex effort costs and it is not something that can typically be found in the dynamic experimentation literature. It shows the complexity of the experimentation rates that we can expect to find in the presence of convex effort costs.

4.5. Comparison of the Contractual Outcomes

It is interesting to compare the outcomes of financing different projects using different contracts. It is obvious that the best contract will stay the best and the renegotiable contract will perform the worst in this environment no matter what. However, the question is how far is the distance

Table II.2. Example Projects

Project	Surplus, $R+E+I$	Costs, c	Discount Rates, r	Prior, α_0
A	20	2	0.05	50%
B	9	3	0.07	80%
C	15	3	0.5	70%
D	10	4	0.7	90%

between different contracts in terms of the social (and the entrepreneur's) surplus. I calculate the expected total value generated by following the terms of the three contracts presented in this paper applied to four different projects described in Table II.2.

Project A promises relatively high return, the monetary costs are relatively low, the discount factor indicates high patience, but the project can be good or bad with a fifty-fifty chance. This project is a high-risk-high-return project with patient players. Project B promises low return, costs are medium, patience is high, and the probability that the project is good is 80%. This is a low-risk-low-return project with patient players. Project C can produce high surplus, the costs are medium, the prior is good at the level of 70%, but the players are impatient. This project can be called medium-risk-medium-return with impatient players. Finally, Project D has low return, high costs, high chance of success, but impatient players. This is a low-risk-low-return project with impatient players. For all the projects, the effort cost function, $f(\hat{\epsilon})$ is assumed to be quadratic,

$$f(\hat{\epsilon}) = \hat{\epsilon}^2.$$

The results of experimenting using different funding contracts are demonstrated in Table II.3. The numbers represent the expected total surplus that the entrepreneur receives. Given that the entrepreneur receives the whole surplus, the total expected surplus and the entrepreneur's portion are equivalent. Numbers in brackets are percentages relative to the benchmark efficient outcome. The columns show the financing outcomes for the projects. The rows indicate the contracts. The efficient contract is included for reference only.

Table II.3. Financing Outcomes by Project and by Contract

Contract	A	B	C	D
Efficient	4.98 (100%)	2.36 (100%)	3.45 (100%)	2.05 (100%)
Best	4.88 (98%)	1.77 (75%)	3.24 (94%)	1.22 (60%)
Conditional	4.79 (96%)	1.25 (53%)	3.03 (88%)	0.77 (38%)
Renegotiable	4.69 (94%)	1.12 (47%)	3.01 (87%)	0.72 (35%)

There are no surprises—the best contract outperformed all the alternative contracts for all the projects, and the conditional contract steadily produced higher surplus than the renegotiable contract. The best contract performed exceptionally well in the cases of Projects A and C, but so did the other contracts, producing more than 85% of the efficient surplus at worst. However, for Projects B and D, the best contract outperformed the alternative contracts significantly, reaching the way to efficiency 22 percentage points closer than the second best alternative. It looks like the structure of the project matters less for the projects with high returns and it becomes very important for the projects with low returns. Practically, it means that it could be advisable to utilize unconditional payment schemes for all projects, but it especially beneficial for the projects that involve relatively low returns.

Another observation is that the difference between the performance of the conditional and the renegotiable contract is low for projects that involve low patience. It is also minimal for the projects with high returns. Practically, it means that if the choice is between having a contract with conditional payments and long term structure and the contract that is renegotiated every period, then the difference between the contractual outcomes is going to be minimal if the parties are impatient or if the project at hand yields relatively high returns. Given that the renegotiable contract is much less complicated to write and follow, it can be more practical in these situations. However, when the patience is high and the returns are relatively low, the renegotiable contract should be avoided.

5. Conclusions

I develop a model of an interaction between the entrepreneur and the angel investor at early stages of a startup development. The model includes convex effort costs, savings, and private valuations of the project. I show that in the equilibrium the experiments are performed at suboptimal rates. Even if the entrepreneur has full bargaining power and is willing to maximize the ex ante total surplus, the impossibility to commit to the efficient experimentation rates ex post undermines all the efforts to achieve efficiency ex ante. The main reasons for inefficiency are liquidity constraints: the entrepreneur needs to receive funds from the external sources. She also needs to compensate the investors by offering them a share in a project surplus, which directly affects the ex post incentives. The projects are risky and the entrepreneur's actions are unobservable: these properties make it impossible to write the contracts contingent on anything other than the event of success. I show that the efficient experimentation rates should decrease in time and that the equilibrium experimentation only stops in the event of success. Otherwise, the experiments continue forever.

I characterize the experimentation paths produced by the best contract that the entrepreneur can propose to the investor and compare the funding outcomes of writing the best contract to the outcomes of funding the projects under two alternative schemes: the conditional funding and the renegotiable contract. I show that the best contract involves no conditional payments and it is especially preferable in the situations when the project involves relatively low returns. When choosing between the other two alternative contracts, the entrepreneur can sometimes prefer the renegotiable contract due to its lack of complexity. It can be preferable in the situations when the parties are impatient or when the project promises relatively high returns.

Dynamics, convex costs, and continuous time are employed to reveal the complexity of the dynamic agency problem. I show that the experimentation paths and the policy function produced in such environments can have surprising properties, like in the case of the renegotiable contract that involves overexperimentation, which is not typically found in the literature with linear effort costs.

Chapter III

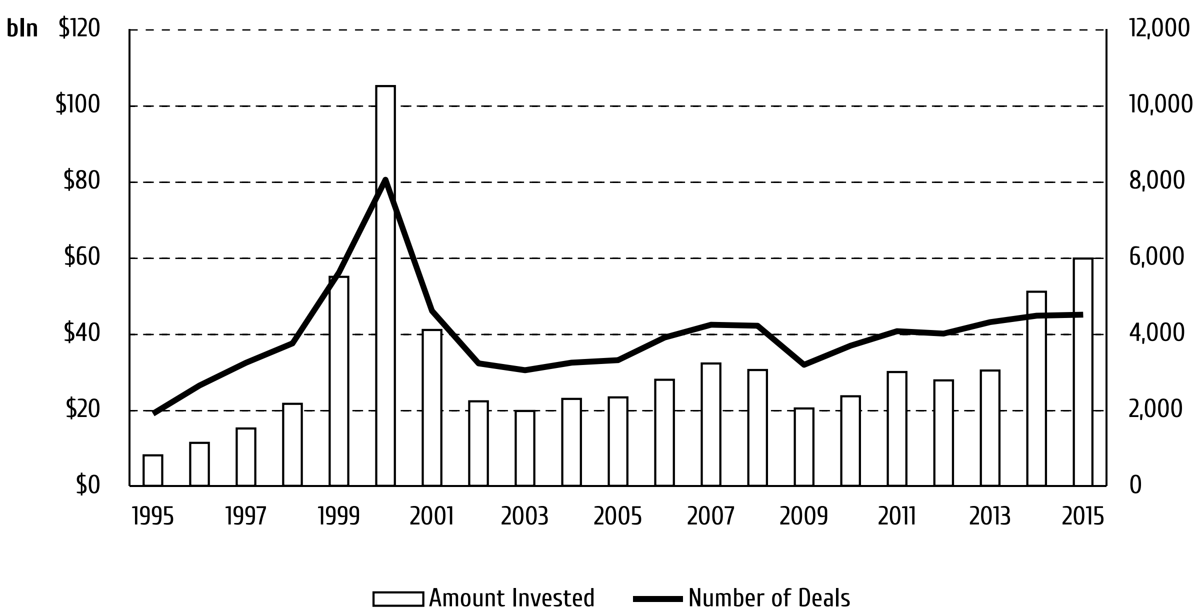
Funding Projects with Diminishing Marginal Returns

1. Introduction

1.1. Relevance of the Problem

The role of venture capital markets in the modern innovation-driven economy is undeniably important. In 2013, venture-capital-backed companies accounted for 43% of all the public companies founded since 1979 in the U.S. (Gornall and Strebulaev, 2015). Among these public

Figure III.1. U.S. Venture Capital Market Size (Data Source: PwC and NVCA, 2016)



companies, venture-capital-backed enterprises also employed 38% of the employees (about 2% of the total workforce), comprised 57% of the market capitalization (13% of the total U.S. market capitalization), and, most importantly, were responsible for 82% of research and development expenditures (21% of the total R&D expenditures in the U.S.). According to KMPG (Fortnum et al., 2016), in 2015, global investments in venture capital amounted to $129.5 billion. In the U.S., this number was almost $60 billion (PwC and NVCA, 2016). The dynamics of investments in venture capital and the total number of deals in the U.S. market are presented in Figure III.1.

Given that venture capital plays such an important role, it is crucial to understand the theoretical aspects of this market. In particular, it is important to know how new venture-capital-backed companies are funded, what their optimal funding schedule should be, why funding may become suboptimal, what affects the funding schedule, and what possible steps can be done to alleviate the inefficiencies if they are present. Answers to these questions will shape the policy decisions and may uncover new trends in the venture capital market. Theoretical models developed to address these issues may be eventually used to analyze the actual behavior of investors and entrepreneurs based on empirical data.

1.2. Research Question

I study a model in which the investor provides funds to the entrepreneur to develop a promising risky project with diminishing marginal returns to effort level. The aim is to characterize the funding schedules that arise under three different scenarios ranked by decreasing amount of observability and verifiability of information. In the first scenario, the entrepreneur's effort levels and investment amounts are fully observable by the investor and verifiable by a third party. In the second scenario, the entrepreneur's actions are still observable by the investor, but are not verifiable by a third party. In the third scenario, the entrepreneur's actions are completely unobservable: it is impossible to tell what the entrepreneur does with the funds provided by the investor.

The main research question is how the funding schedule depends on the information available to the investor and on the parameters of the model, which include the total size of the surplus the project generates upon success, the patience of the players represented by the discount factor, their initial optimism about the project, and the marginal monetary costs of conducting experiments.

1.3. Related Literature

The model I describe borrows from and extends upon several well-known and acclaimed papers that together contribute to the body of dynamic agency and venture capital research. The literature on this topic is rich, and there are many important models, so the reference list I include in this chapter covers only the most relevant literature, but does not list all the papers that were developed over the years. For the more detailed history of the dynamic agency models and bandit games in general, see Bergemann and Välimäki (2008).

The model by Bergemann and Hege (2005) was among the first to describe the interaction between the venture capitalist and the entrepreneur using the bandit model framework. They assumed that there is an entrepreneur who has a project, which may or may not succeed in the future, but does not have capital to finance the actual research on the project. Every period, the entrepreneur asks for the funds in exchange for a share of the possible surplus, and the investor provides funds to the entrepreneur to conduct experiments. Experimenting is needed to find out if the project is viable. In their paper, probability that the project succeeds linearly depends on the amount of money spent on experiments. The main problem they specifically address is that since the investor does not see what the entrepreneur is doing with the money, the entrepreneur may decide to divert the funds and pretend to conduct experiments while actually shirking. The authors showed that under certain parameter values the model may predict that the funds will be provided at a constant rate, at a decreased rate ("frontloading"), or at an increased rate ("backloading"), but in any case, the funding schedule will be suboptimal.

Hörner and Samuelson (2013) later confirmed the findings of Bergemann and Hege (2005) by analyzing essentially the same model in continuous time, but with the full bargaining power given to the investor. They assumed that the probability of success of each experiment is given, and the entrepreneur can only influence the continuous experimentation rate by delaying the experimentation. Their model revealed that the funding rate may increase or decrease in time under Markovian decision-making assumption, and this fact prompted them to consider alternative interpretations of the model. One of the subsections of the paper specifically addresses Markov equilibrium issues and proposes that non-Markovian strategies are a better alternative. In fact, they state: "we believe that non-Markov equilibria better reflect, for example, actual venture capital contracts."[1] Indeed, their non-Markovian equilibria exhibit the intuitive "frontloading" feature that we expect to find in the real world.

This chapter addresses the problem differently. I keep the simplicity, convenience, and empirical appeal of the Markovian strategic environment, but instead of the linear effort costs used in the previous papers, I assume convex per-period effort costs. This assumption is realistic because in real-world environments improving the probability of success in a particular experiment is increasingly hard, while reaching the probability of one (conditional on the project being good) is usually prohibitively expensive. This critical modeling adjustment not only changes the outcomes of the analysis, but it also requires implementing different solution techniques that I develop here as well.

There are not many papers that consider convex cost environments in the dynamic agency settings. Mason and Välimäki (2015) analyze the behavior of the dynamic agency model in the presence of convex costs under the assumption that parties have dissimilar discount rates, but they only consider projects that are certain to succeed. Another model that also analyzes the dynamic moral hazard problem with convex effort costs is by Bhaskar (2013). He studies the optimal contracting scheme between a risk-neutral principal and a risk-averse agent in the presence of public signals and unobserved actions in two time periods. His paper also contains a discussion about the problems that arise in the environments in which players make both continuous and discrete choices and what role indifference has to play in such problems.

1.4. Main Contribution

The contribution of this chapter is twofold. First, I demonstrate that in the presence of the convex effort costs assumption, Markov strategies produce the desired and intuitive "frontloading" funding schedules in full information and observable but unverifiable information environments. In the unobservable information environment, the funding rates may increase in the beginning for

[1] Hörner and Samuelson (2013), page. 634.

some promising projects for a short time, but eventually the funding rates will become strictly decreasing. This is both similar to and different from the results of Hörner and Samuelson (2013): I show that for some projects, especially for the high surplus and impatient projects, the funding rates may increase in the beginning if the parties are optimistic enough; however, I demonstrate that the funding rates quickly become decreasing, so there is no strict "backloading" in this model.

Second, I develop a methodology of solving the dynamic agency models in continuous time in the presence of the convex effort costs for the first-best case and for the two different equilibria cases. The manipulation with value functions developed here will be useful for possible extensions of this model, some of which are also discussed.

1.5. Additional Results

In addition to the main results, I reveal that the simulations strongly suggest that the investors will prefer the completely unobservable environment to the environment with observable but unverifiable actions under the Markovian assumption. It means that if the investor can commit not to monitor the activity of the entrepreneur, then he will be better off by not knowing what the entrepreneur is doing with the money. This is because the unobservability strengthens commitment: if the entrepreneur diverts the funds and the investor does not believe her, then he will still provide the funds according to the equilibrium funding scheme. If, however, the diversion by the entrepreneur is observable, then the investor will know that the experimentation did not happen at the desired level, and so he will be more optimistic about the project than if he believed that the funds were actually spent on experiments. The unobservability makes the threats of diverting the funds by the entrepreneur less credible.

I also characterize how the funding schedules depend on the parameters of the model. In all the scenarios, being impatient is beneficial to both parties because it promotes early effort application and discounts future experimentation opportunities heavily. The surplus to marginal monetary costs ratio is another important driver of the funding rates. Not only does it positively directly affects the funding rates at each time period, but it also allows projects with lower chances of success to be funded: critical optimism level required to finance a project decreases in the surplus to marginal costs ratio.

1.6. Structure of the Chapter

The chapter continues with the description of the model, including the characterization of the equilibrium concept and the strategies of the players. Then I move to describe the behavior of the funding schedule in the first best scenario and how it is implementable in this model. What follows in the characterization of the funding timeline in two equilibria cases. After that, I characterize the comparative statics based on simulations and describe possible viable extensions of the model.

2. The Model

2.1. Players

The model is initially constructed in discrete time, and then it is extended to the continuous time environment. I assume that the minimal length between any two decision points is the same and equal to $d > 0$. Essentially, I analyze the limiting case, when $d \to 0$. This technique ignores some features of the continuous time that suggest a wider range of possible behavioral patterns of the players, but since I am analyzing the Markov equilibria, this technique fits the research program well.

The model is standard for the dynamic agency literature. There is an entrepreneur who has a project. The entrepreneur believes that the project is good with the prior probability equal to α_0. To complete the project, the entrepreneur needs to conduct experiments—risky operations that, conditional on the project being good, may lead to a success with probability $\epsilon_t d$ at a time period, t, of length d or to a failure with the complement probability. If the experiment is successful, the project is complete and it generates the surplus, R. If it is not successful, the entrepreneur may decide to continue experimenting in the next period.

The biggest problem here is that bad projects never succeed, no matter how hard the entrepreneur tries and how many experiments she conducts. Therefore there is always a sense of doubt when the experiment does not succeed: is it just because this attempt was unsuccessful, or is it because the project is ultimately bad?

Each time period, t, the entrepreneur can only conduct a single experiment. After each experimentation attempt, the entrepreneur updates her belief that the project is still good by the means of the Bayes' rule:

$$\alpha_{t+d} = \frac{\alpha_t (1 - \epsilon_t d)}{1 - \alpha_t + \alpha_t (1 - \epsilon_t d)} = \frac{\alpha_t (1 - \epsilon_t d)}{1 - \alpha_t \epsilon_t d}.$$

This posterior belief, α_t, can only decrease over time, reflecting the fact that the entrepreneur becomes more and more pessimistic about the success chances of the project. The decline is stronger the larger the ϵ_t is, indicating the trade-off between the higher chances of succeeding in the current period or being more optimistic (and more incentivized) in the future.

The probability that the experiment is a success after exactly n trials, where $n \in \mathbb{N}$, is

$$P(X = n) = \alpha_{nd} \epsilon_{nd} d \prod_{i=1}^{n-1} (1 - \alpha_{id} \epsilon_{id} d).$$

The distribution of random variable X resembles geometric distribution, however, it is different since the probability of success per trial, $\alpha_{id}\epsilon_{id}$, may change between trials. Also, the probability that the project never succeeds may be strictly positive for some projects:

$$P(X = \infty) = 1 - \alpha_0 \geqslant 0.$$

To conduct a single experiment, the entrepreneur has to exercise some effort and spend some money. However, she has no funds of her own, so she seeks to obtain the external funding. In this model, financing is available only from a single investor who shares the prior belief, α_0, about the chances that the project is good with the entrepreneur.

Each time period t, the investor offers to invest $\gamma_t c$ in the project in exchange for share $(1 - s_t)$ of the surplus, R, which he will receive only if the project succeeds. Alternatively, the investor can offer a complete contract that describes the funding path and the shares for each period, but without any commitment assumptions, such a contract will perform just as well as no contract at all, hence, it is not required.

Once the entrepreneur obtains the money, she can either spend it on an experimentation attempt or divert the funds to her private needs. Depending on the scenario, the investor may or may not observe how the money is spent, so every time he invests, there may be uncertainty about whether the money he had given to the entrepreneur was actually spent experimenting.

2.2. Renegotiation and Commitment

I assume that the players will not be able to commit to any contract that presupposes a possible waste of the expected surplus. In particular, it will be hard to commit to stop providing funds at time T if both players believe that it may be beneficial to continue funding the project beyond T given it would not have succeeded by that time. It will also be impossible to implement contracts with third-party payments as clearly it will be a potentially wasteful contract that both players will want to renegotiate shall they actually reach the stage when they need to make side payments. Finally, it will be impossible to commit not to have any relationship after a certain event for either party because if there is a potential surplus to share, then ex post, it will be reasonable to continue with the experimentation.

The players will not commit to stop at time T. Committing to a particular stopping time, T, may be a good idea to enforce certain experimentation rates because the agent will not expect to receive any surplus if the project is not a success until time T. Thus the entrepreneur will try harder to reach a success because there will be less opportunities to reach success in the future. However, if the project is still not a success at time T and the parties are still optimistic enough, then it will be reasonable for them to renegotiate and continues with the experimentation. Thus, the commitment to a stopping time is not viable.

The players will not write contracts that presuppose third-party payments. The mechanisms with stage games and third party payments are good tools to overcome the problems that arise in the environments with observable but unverifiable information (see Moore and Repullo, 1988). The strength of these mechanisms is based on the fact that the parties ex ante agree to pay a penalty to a third party if they disagree about their vision of the state the project ex post. Thus, the mechanisms enforce truth-telling and commitment to the terms of the contract. Unfortunately, such mechanisms are not "renegotiation proof" (see Hart and Moore, 1988), because if the parties actually reach the stage when they need to make the third-party payments, they will want to renegotiate and split the payments between themselves, as paying to a third party is a waste of surplus. Hence, the mechanism breaks down and the parties will not be able to commit to a subgame-perfect implementation with the third-party payments.

It will also be very hard for the parties to commit to break up after a specific event for the same reasons it will be hard to commit to a certain stopping time: as long as there is a surplus to share, the parties are better off working on the project together.

2.3. Information Environments

I consider three environments based on the amount of information available to the investor and to the public (or a third party). The simplest case is the perfect observability of the entrepreneur's actions by the investor and their perfect verifiability by a third party, which may be a judge or a jury. It essentially means that the venture capitalist can write a contract contingent on the entrepreneur's actions, and shall there be a disagreement between the two, a judge or a jury will be able to verify if the action had been indeed performed and what exactly had been done. It also means that the entrepreneur and the investor share the beliefs about the project's viability and update their beliefs based on the same information. So they not only have a common prior, but they also share a common posterior at each period in time.

The second information environment of interest is the case when the entrepreneur's actions are *observable* by the investor, but *unverifiable* by a third party. Such a situation results in the practical impossibility of writing contracts contingent on the entrepreneur's actions: shall the dispute between the principal and the agent ensue, a judge or a jury will be unable to resolve it since they cannot verify claims regarding the effort of the entrepreneur. This happens due to the private nature of the relationship between the principal and the agent: even if the information regarding the effort level becomes available to a third party, it will be provided by either the principal or the agent. Hence it may be biased and not reflect the actual state of the world. However, both the principal and the agent are equally informed about the state of the world, and so the principal can make offers based on the commonly shared posterior belief that the project is still good. It also potentially allows to implement the first best *informally* under the threat of reverting to the suboptimal funding rates. The environments with observable but unverifiable states have been studied extensively by Hart and Moore (1988), Green and Laffont (1988), Maskin and Moore (1999), and many others. A traditional solution to overcome the unverifiability involves truth-telling mechanisms with side payments based on the mechanism by Moore and Repullo

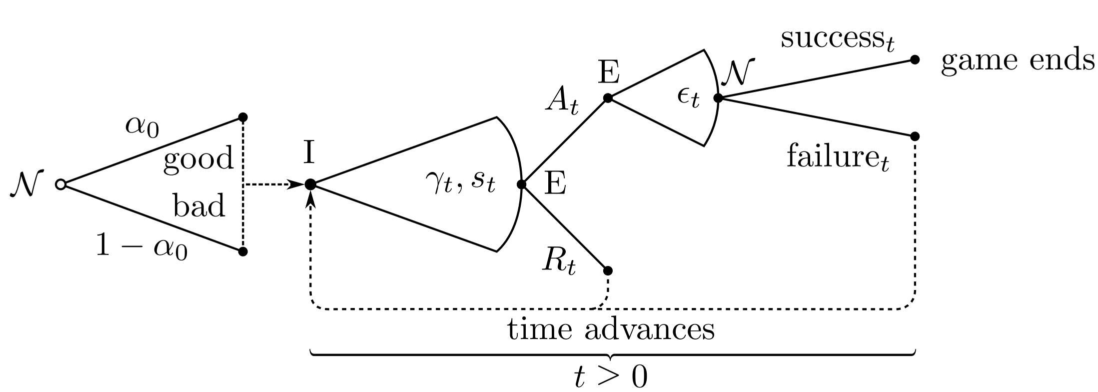
(1988) that have a big problem of being non-renegotiation proof in general. Therefore, I do not consider these mechanisms in this chapter.

The third, and the final environment I consider is the complete unobservability of the actions. The only event that the investor can observe is the event of the success of the project. In such an environment, not only action-contingent contracts are impossible, but the investor literally has no idea what the entrepreneur is doing with the money, hence he can only provide the incentives for the entrepreneur to actually experiment and hope that these incentives are sufficient. It means that the entrepreneur updates her posterior beliefs that the project is good based on the actual effort level, and the investor updates his posterior based only on what he believes had been invested. Any deviation by the entrepreneur in such an environment is unobservable, and so no punishment for not following the contract is possible.

2.4. Game Timing and Actions

The game is pictured in Figure III.2. At the beginning of every period t, the investor (I) offers the entrepreneur (E) a share, s_t, and a sum of money, $\gamma_t c$. The share, s_t, is the share of the surplus, R, that will be allocated to the entrepreneur if the project succeeds in period t, the investor gets the rest. Every time period, the share and the sum of funds are renegotiated. The agreement is for the current time period only. Next, the entrepreneur decides whether to accept or reject the proposal. If she accepts, she then decides how much effort to exercise to conduct an experiment.

Figure III.2. Game Scheme



Whatever money is not spent experimenting, gets diverted. If she rejects the proposal or if the experimentation session fails, time advances to the next period, everyone updates their beliefs that the project is still good, and everything starts over. The game ends if the project succeeds or if the investor stops providing funds indefinitely.

2.5. **Equilibrium Concept**

The equilibrium concept is Markov sequential in pure strategies. In particular, it means that the players' strategies are based on their current beliefs about the state of the world. This way, the past history of the players' actions is aggregated in a certain way to characterize the current state. There may exist different interpretations of what it actually entails, so it must be specified explicitly.

The investor's offer of s_t and γ_t is based only on his belief that the project is still good, $\hat{\alpha}_t$. Therefore the investor's strategy is just

$$\hat{\alpha}_t \mapsto \left(s\left(\hat{\alpha}_t\right), \gamma\left(\hat{\alpha}_t\right)\right)$$

defined for all possible beliefs about the state, $\hat{\alpha}_t$. The investor updates his current beliefs about the state based on his beliefs about the effort level the entrepreneur exercised in the previous period and the previous period's beliefs about the state of the world:

$$\hat{\alpha}_{t+d} = \frac{\hat{\alpha}_t\left(1-\hat{\epsilon}_t d\right)}{1-\hat{\alpha}_t\hat{\epsilon}_t d}.$$

This transition equation describes the evolution of state variable $\hat{\alpha}_t$.

Along the equilibrium path, this belief is consistent with the actual probability that the project is good. Given that the investor can never tell if the deviation on the entrepreneur's side occurred, the investor always believes that the entrepreneur's posterior is $\hat{\alpha}_t$.

The entrepreneur's state is more complicated. First, the entrepreneur knows the true posterior probability that the project is good, α_t. Second, she also knows what the investor believes the posterior belief is, $\hat{\alpha}_t$, because they share the same prior and because she knows how the investor forms beliefs. Third, she observes the current offer of s_t and γ_t, so the entrepreneur's decision regarding the effort level is based on three components:

$$\alpha_t, \hat{\alpha}_t,\ \left(s_t, \gamma_t\right) \mapsto \epsilon\left(\alpha_t, \hat{\alpha}_t,\ \left(s_t, \gamma_t\right)\right).$$

In the environments where the entrepreneur's actions are observable, $\hat{\alpha}_t = \alpha_t$, and so the entrepreneur's strategy is less complicated:

$$\alpha_t,\ (s_t, \gamma_t) \mapsto \epsilon\left(\alpha_t,\ (s_t, \gamma_t)\right).$$

The reasons to include the current period offers as the components of the entrepreneur's space set are dictated by the requirement of the sequential rationality: the actual experimentation happens after the offer is accepted, so it can depend on the offers, as well as on the state variables.

A deviation in such a setting is simply a change in the way the players respond to the state of the world. A one-shot deviation corresponds to deviating in response to one particular state and leaving the rest of the strategy intact. For example, the entrepreneur may deviate at some particular time t from the equilibrium strategy, $\epsilon^{**}\left(\alpha_t, \hat{\alpha}_t,\ (s_t, \gamma_t)\right)$, but then return to the equilibrium play consistent with the new state variables' evolution paths created by the deviation. In particular, it means that once the entrepreneur deviated, she may begin diverting some of the funds to her own needs forever, since the path of the state variables will be different from the equilibrium path.

The one-shot deviation concept is complicated in continuous time as a payoff change at one particular instant has no effect on the total payoff function, and yet Hamiltonian-Jacobi-Bellman equations allow one to treat these seemingly unimportant deviations as critical by concentrating on particular instances of time, so it is possible to construct the best response functions pointwise while keeping the rest of the strategies fixed.

Given the Markovian nature of the decision process, the offer acceptance stage of the game is redundant: if the entrepreneur rejects the offer, the investor will credibly believe that nothing would be invested, so he will update his believes about the state according to $\hat{\alpha}_{t+d} = \hat{\alpha}_t$ and will make the same offer at time $t + d$ as he had made at time t. From the entrepreneur's perspective, it means that she's just wasted one period disagreeing and still got the same offer.

The problem of the uniqueness of the equilibria I describe below has not been addressed yet. There may be additional equilibria in the environment with the unobservable actions.

2.6. Payoffs

The interim utility functions are simple for both the principal and the agent. Experimentation requires both effort and money, so a total cost to conduct a single experiment at time t is

$$\epsilon_t dc + f_d(\epsilon_t) d,$$

where $\epsilon_t dc$ is the monetary part of the cost (c being the marginal monetary cost) and $f_d(\epsilon_t) d$ is the cost of effort. Alternatively, there could be just one cost function, but I specifically keep this form to be able to compare the results of my analysis to the ones obtained by Hörner and Samuelson (2013) and Bergemann and Hege (2005). It also allows seeing how much has been diverted.

The effort cost is represented by function f_d, which is continuous, at least thrice differentiable, and strictly convex and increasing, with $f_d(0) = 0$. So every time the entrepreneur wishes to exercise some effort $\epsilon > 0$, it will cost her $f_d(\epsilon) d > 0$. One convenient property beside convexity that I want to have is $f_d'(0) = 0$,[2] but it is not required. The second property I need to state explicitly is $f_d'''(x) \geqslant 0$ for any $x \in \left[0, \frac{1}{d}\right]$.

There are several reasons to consider the convexity of the effort costs. First, it provides simple and intuitive way to limit probability of success below one: the traditional models have used the linear costs and so the authors had to introduce caps to the effort level. Second, this assumption is natural: diminishing marginal returns are at the heart of many economic phenomena including the experimentation outcomes. Even if the cost curve is "S"-shaped, still, we will be interested in the part where additional effort brings less and less chances of success. Third, the traditional approach to solving these models required considering trigger strategies that have a problem of introducing the indifference, which means that mixed strategies become possible, as well as the fact that the entrepreneur is indifferent between experimenting and diverting all the funds. It makes the equilibria fragile and less realistic. Convex costs ensure that only a particular effort

[2] Here, and everywhere in the text, for some function $g(x)$, $g'(x)$ means first derivative, $g''(x)$—second, and so on. For some x, $\dot{x}$ indicates differentiation with respect to time, or $\frac{\mathrm{d}x}{\mathrm{d}t}$. For some function $h(x, y)$, $h_1(x, y)$ means partial derivative with respect to the first variable, $h_2(x, y)$—partial with respect to the second variable, $h_{12}(x, y)$—partial derivative with respect to the first, and then to the second variable, and so on.

level is optimal, other parameters fixed. Fourth, strategic experimentation models are solved in discrete time, and then the continuous-time solution is produced as a limiting case, when the length of the time between two periods becomes infinitely small. Using the convex costs and Hamilton-Jacobi-Bellman equations, I can solve continuous time models directly and produce the same outcomes as if I used the limiting results obtained in the discrete case.

The entrepreneur's benefit from experimenting in the current period is simply

$$\alpha_t \epsilon_t d s_t R + \gamma_t dc,$$

where $\alpha_t \epsilon_t d$ represents the probability that the project will succeed in the current period given the effort level of $\epsilon_t d$ and posterior probability that it is good α_t; $s_t R$ is the share of the surplus the entrepreneur will get if the project is successful in the current period. Note that the benefit includes $\gamma_t dc$—the funds provided by the investor. This is because the entrepreneur does not pay for the experimentation out of her pocket, and so this advance is clearly her income. In the end, the difference between the funds she received and the funds she actually spent experimenting,

$$\gamma_t dc - \epsilon_t dc,$$

determines how much she diverted towards her personal consumption.

The venture capitalist provides funds to the entrepreneur. Alternatively, he could have used the funds for his personal needs. Therefore, the cost of providing the funds is simply

$$\gamma_t cd.$$

The investor's benefit is just

$$\hat{\alpha}_t \hat{\epsilon}_t d \left(1 - s_t\right) R,$$

where $\hat{\alpha}_t \hat{\epsilon}_t d$ is the investor belief that the project will succeed at time t, and $\left(1 - s_t\right) R$ is the share of the surplus he would get if the outcome of the experimentation is favorable.

2.7. Players' Problems

Every time period, after accepting the offer of γ_t and s_t, the entrepreneur needs to decide upon the effort level, ϵ_t. Given a common discount factor of δ and the probability that the project will fail in the current period $(1-\alpha_t\epsilon_t d)$, the entrepreneur's problem in a recursive form is

$$V(\alpha_t, \hat{\alpha}_t) = \max_{\epsilon_t} \left[\alpha_t\epsilon_t d s_t R - f_d(\epsilon_t) d + \gamma_t dc - \epsilon_t dc + (1-\alpha_t\epsilon_t d)\,\delta V(\alpha_{t+d}, \hat{\alpha}_{t+d})\right]$$

subject to the participation constraint,

$$V(\alpha_t, \hat{\alpha}_t) \geqslant 0,$$

$$\epsilon_t \leqslant \gamma_t,$$

and Bayesian updating of posterior beliefs α_t and $\hat{\alpha}_t$.

For the investor, the problem is a bit more complicated. He needs to make an offer of γ_t and s_t to provide sufficient incentives for the agent to invest exactly as much as he believes she will invest. If he believes that the experiment will fail with probability $(1-\hat{\alpha}_t\hat{\epsilon}_t d)$, then his problem is

$$W(\hat{\alpha}_t) = \max_{s_t,\,\gamma_t} \left[\hat{\alpha}_t\hat{\epsilon}_t d(1-s_t) R - \gamma_t dc + (1-\hat{\alpha}_t\hat{\epsilon}_t d)\,\delta W(\hat{\alpha}_{t+d})\right]$$

subject to the participation constraint,

$$W(\hat{\alpha}_t) \geqslant 0,$$

and the incentive constraint,

$$\hat{\epsilon}_t = \arg\max_{\epsilon_t} \left[\hat{\alpha}_t\epsilon_t d s_t R - f_d(\epsilon_t) d + \gamma_t dc - \epsilon_t dc + (1-\hat{\alpha}_t\epsilon_t d)\,\delta V(\hat{\alpha}_{t+d}, \hat{\alpha}_{t+d})\right]$$

subject to

$$\hat{\epsilon}_t \leqslant \gamma_t,$$

and the evolution of posterior beliefs $\hat{\alpha}$. The incentive constraint indicates that along the equilibrium path the investor forms rational beliefs about the future actions of the entrepreneur.

A consistent solution to these two problems together comprises an equilibrium because every player best-responds to the other player's action. The equilibrium is of the form of the policy functions

$$\gamma(\hat{\alpha}),\ s(\hat{\alpha}),\ \text{and}\ \epsilon(\alpha, \hat{\alpha}),$$

and consistent equilibrium beliefs of the investor, $\hat{\epsilon}(\hat{\alpha}) = \epsilon(\hat{\alpha}, \hat{\alpha})$. Off-equilibrium beliefs are only required for the unobserved action environment and are simple: since the deviation by the entrepreneur is non-detectable *ex post* and no information escapes the entrepreneur, then the investor simply has the same belief for any possible deviation the entrepreneur can make.

2.8. From Discrete to Continuous Time

The process of transforming the problem in discrete time to the the problem in continuous time is described in Appendix III.A. I keep all the variable names similar to their discrete-time counterparts, though their meaning changes sometimes. For example, $\epsilon_t d$ used to mean probability of success of a single experiment, now ϵ_t means the rate of effort, and as such, it is not bounded from above (yet it still must be finite). Similarly, function f_d had a property that $f_d(x) \to \infty$ as $x \to \frac{1}{d}$, but as $d \to 0$, function f_d converged to f that is still strictly increasing, convex, and satisfies all the other properties of f_d, but now it only needs to converge to infinity as the effort level goes to infinity.

The entrepreneur's problem in continuous time can now be written as a Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rV(\alpha, \hat{\alpha}) = \max_{\epsilon} [&\alpha\epsilon sR - f(\epsilon) + \gamma c - \epsilon c - \alpha\epsilon(V(\alpha, \hat{\alpha}) + (1-\alpha)V_1(\alpha, \hat{\alpha})) \\ &- \hat{\alpha}\hat{\epsilon}(1-\hat{\alpha})V_2(\alpha, \hat{\alpha})] \end{aligned} \tag{III.1}$$

subject to the participation constraint:

$$V(\alpha, \hat{\alpha}) \geqslant 0$$

and

$$\epsilon \leqslant \gamma.$$

This is the continuous-time analogue of the Bellman equation, and it behaves similarly: it is assumed that at every instance, the future path is optimal given the state variable, and so the only thing left to do is to find the best behavior at every instance given the current state and the influence current decision will have on the future optimal path. HJB equations allow to treat instantaneous deviations in continuous time as if they were potentially profitable.

In the similar fashion, the investor's problem in continuous time is

$$rW\left(\hat{\alpha}\right) = \max_{s,\gamma}\left[\hat{\alpha}\hat{\epsilon}\left(1-s\right)R - \gamma c - \hat{\alpha}\hat{\epsilon}\left(W\left(\hat{\alpha}\right) + \left(1-\hat{\alpha}\right)W'\left(\hat{\alpha}\right)\right)\right] \tag{III.2}$$

subject to the participation constraint,

$$W\left(\hat{\alpha}\right) \geqslant 0,$$

and the incentive constraint,

$$\hat{\epsilon} = \arg\max_{\epsilon}\left[\hat{\alpha}\epsilon sR - f\left(\epsilon\right) + \gamma c - \epsilon c - \hat{\alpha}\epsilon\left(V\left(\hat{\alpha},\hat{\alpha}\right) + \left(1-\hat{\alpha}\right)V_1\left(\hat{\alpha},\hat{\alpha}\right)\right)\right.$$
$$\left. - \hat{\alpha}\hat{\epsilon}\left(1-\hat{\alpha}\right)V_2\left(\hat{\alpha},\hat{\alpha}\right)\right],$$

$$\hat{\epsilon} \leqslant \gamma.$$

It is useful to have the expressions of both the value functions at time t in the infinite integral representation form. For the entrepreneur, it becomes

$$\begin{aligned} V\left(\alpha_t,\hat{\alpha}_t\right) = \int_t^{\infty} e^{-(\tau-t)r-\int_t^{\tau}\alpha_\theta\epsilon(\alpha_\theta,\hat{\alpha}_\theta)\,\mathrm{d}\theta}\left[\alpha_\tau\epsilon\left(\alpha_\tau,\hat{\alpha}_\tau\right)s\left(\hat{\alpha}_\tau\right)R - f\left(\epsilon\left(\alpha_\tau,\hat{\alpha}_\tau\right)\right)\right. \\ \left. + \gamma\left(\hat{\alpha}_\tau\right)c - \epsilon\left(\alpha_\tau,\hat{\alpha}_\tau\right)c\right]\,\mathrm{d}\tau. \end{aligned} \tag{III.3}$$

For the investor, it is a bit simpler:

$$W\left(\hat{\alpha}_t\right) = \int_t^{\infty} e^{-(\tau-t)r-\int_t^{\tau}\hat{\alpha}_\theta\hat{\epsilon}(\hat{\alpha}_\theta)\,\mathrm{d}\theta}\left[\hat{\alpha}_\tau\hat{\epsilon}\left(\hat{\alpha}_\tau\right)\left(1-s\left(\hat{\alpha}_\tau\right)\right)R - \gamma\left(\hat{\alpha}_\tau\right)c\right]\,\mathrm{d}\tau. \tag{III.4}$$

Functions $\epsilon\left(\alpha_t,\hat{\alpha}_t\right)$, $s\left(\hat{\alpha}_t\right)$, and $\gamma\left(\hat{\alpha}_t\right)$ are the policy functions, and $\hat{\epsilon}\left(\hat{\alpha}_\tau\right) = \epsilon\left(\hat{\alpha}_t,\hat{\alpha}_t\right)$.

Depending on the information environment at hand, these problems, value functions, and policy functions will be adjusted. In the current formulation, they reflect the worst-case scenario, in which the investor does not observe the entrepreneur's actions, and in this sense, this is the most general form of the problems that are solved below.

3. The First Best

3.1. Description

The first-best scenario is straightforward. Both parties observe the effort level produced by the entrepreneur and this information is easily verifiable in the court. This implies the possibility to make offers contingent on the effort level, not just on the event of the success of the project. I will first describe the social planner's solution to this problem and then I will discuss how to implement the first best funding scheme in the actual full-information environment.

Appendix III.C is devoted to deriving of the combined value function and other essential steps of the solution. Since the utilities are transferable, combining the value functions under the assumption that there is no disagreement regarding the state of the world (represented by the posterior belief that the project is still good, α_t, at every time instance t) will produce a value function that can be manipulated directly from the perspectives of the social planner.

The social planner's problem is simply

$$r\mathcal{V}\left(\alpha\right)=\max_{\epsilon}\left[\alpha\epsilon R-f\left(\epsilon\right)-\epsilon c-\alpha\epsilon\left(\mathcal{V}\left(\alpha\right)+\left(1-\alpha\right)\mathcal{V}'\left(\alpha\right)\right)\right].$$

Participation constraint is not required: setting $\epsilon\left(\alpha\right)=0$ everywhere guarantees that the value function is at least zero.

The first order condition is

$$\alpha R-f'\left(\epsilon^{*}\left(\alpha\right)\right)-c-\alpha\left[\mathcal{V}'\left(\alpha\right)\left(1-\alpha\right)+\mathcal{V}\left(\alpha\right)\right]=0,$$

or

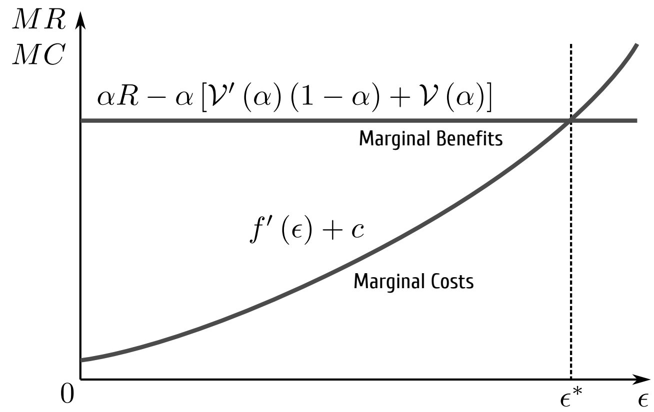
$$f'\left(\epsilon^{*}\left(\alpha\right)\right)+c=\alpha R-\alpha\left[\mathcal{V}'\left(\alpha\right)\left(1-\alpha\right)+\mathcal{V}\left(\alpha\right)\right].$$

The second-order condition is satisfied always. Function $\epsilon^{*}\left(\alpha\right)$ is the first-best policy function. Each time instance, the process of finding the optimal funding rate is presented in Figure III.3. The interpretation is simple: $f'\left(\epsilon\right)+c$ is the marginal cost of applying the effort, αR is the immediate marginal benefit, while $\alpha\left[\mathcal{V}\left(\alpha\right)+\left(1-\alpha\right)\mathcal{V}'\left(\alpha\right)\right]$ is the opportunity cost representing the trade off between the higher possible success rate now and the resulting increased pessimism tomorrow. Similar patterns arise in the other information environments.

Multiplying both sides of the first order condition by $\epsilon^{*}\left(\alpha\right)$ and combining it with the HJB equation yields

$$r\mathcal{V}\left(\alpha\right)=f'\left(\epsilon^{*}\left(\alpha\right)\right)\epsilon^{*}\left(\alpha\right)-f\left(\epsilon^{*}\left(\alpha\right)\right). \tag{III.5}$$

Figure III.3. Instantaneous Efficient Funding Rate



Differentiating the both sides with respect to α produces

$$r\mathcal{V}'(\alpha) = f''(\epsilon^*(\alpha))\,\epsilon^*(\alpha)\,\epsilon^{*\prime}(\alpha)\,. \tag{III.6}$$

These are important equilibrium conditions that together with the first order condition are sufficient to characterize the efficient funding path.

3.2. Main Results

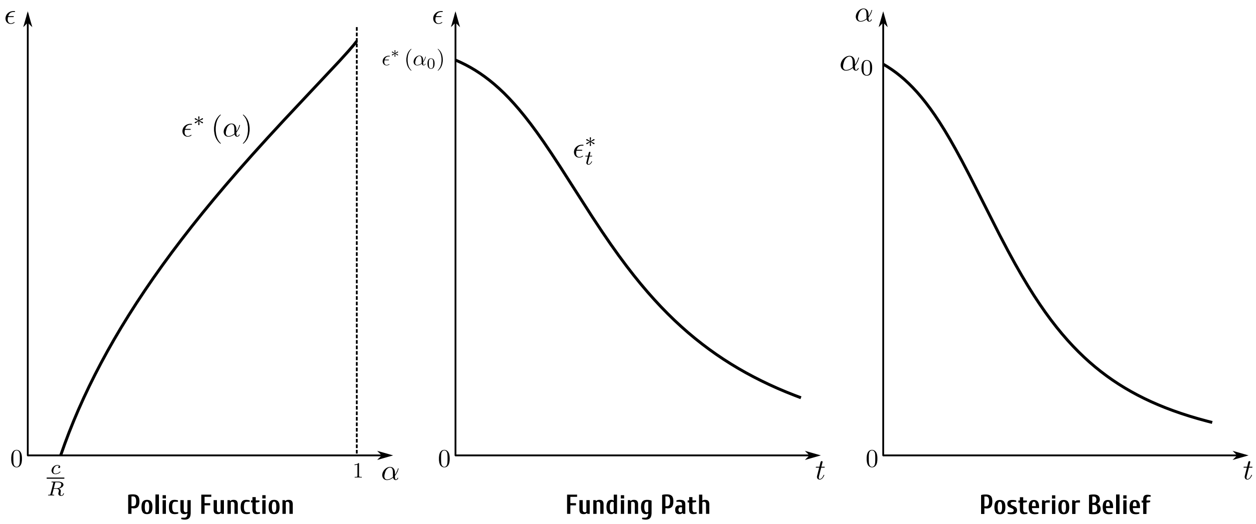
The main results of analyzing the first best scenario are summarized in Proposition III.1 and pictured in Figure III.4.

Proposition III.1. *If the prior belief that the project is good, α_0, is in $\left(\frac{c}{R},1\right)$, and if the effort cost function is strictly convex, then the efficient funding rate strictly decreases over time until the project succeeds or indefinitely. Sure projects with the prior of 1 are funded at a constant rate until success happens. Projects with the prior lower than $\frac{c}{R}$ are never funded.*

Proof. The intermediate components of the proof are located in Appendix III.C. Here, only the intuition is described. First, I show that the efficient policy function $\epsilon^*(\alpha)$ is bounded and strictly increasing.

It is easy to conclude that if $\alpha \leqslant \frac{c}{R}$, then there is no reason to fund the project as even the intermediate benefit becomes lower than the intermediate cost for any positive effort level every

Figure III.4. The First Best Solution



instance of time t. So for all $\alpha \leqslant \frac{c}{R}$, $\epsilon^*(\alpha) = 0$, and for all $\alpha > \frac{c}{R}$, $\epsilon^*(\alpha) > 0$ because the project is still promising.

To see that $\epsilon^*(\alpha)$ is strictly increasing for all $\alpha > \frac{c}{R}$, simply observe that if $\mathcal{V}'(\alpha) > 0$, then

$$r\mathcal{V}'(\alpha) = f''(\epsilon^*(\alpha))\,\epsilon^*(\alpha)\,\epsilon^{*\prime}(\alpha) > 0,$$

which is only possible when

$$\epsilon^{*\prime}(\alpha) > 0.$$

The intuition for why $\mathcal{V}'(\alpha) > 0$ is simple: for any $\alpha`_t > \alpha_t$, there exist some (possibly suboptimal) effort level $\epsilon`_t$ that guarantees that just following $\epsilon`_t$ provides higher social utility than $\mathcal{V}(\alpha)$.

To show that the policy function is bounded, observe that it is definitely bounded from below:

$$\epsilon^*\left(\frac{c}{R}\right) = 0$$

because when $\alpha = \frac{c}{R}$, there is no reason to fund experiments as potential benefits

$$\alpha\epsilon R = \frac{c}{R}\epsilon R = \epsilon c$$

will not worth the costs:

$$\epsilon c \leqslant f(\epsilon) + \epsilon c.$$

To see that the funding rate is bounded from above, combine (III.5) and (III.6) with the first order condition and produce an ordinary differential equation,

$$r\left(\alpha R - f'\left(\epsilon^*\left(\alpha\right)\right) - c\right) - \alpha\left[f''\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right)\epsilon^{*\prime}\left(\alpha\right)\left(1-\alpha\right) + f'\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right)\right.$$
$$\left. - f\left(\epsilon^*\left(\alpha\right)\right)\right] = 0.$$

This ODE produces the efficient policy function, $\epsilon^*\left(\alpha\right)$, when solved given some particular function $f\left(\cdot\right)$ and the boundary condition,

$$\epsilon^*\left(\frac{c}{R}\right) = 0.$$

Consider $\alpha = 1$, the highest possible probability that the project is good, the sure case. Then the ODE becomes

$$f'\left(\epsilon^*\left(1\right)\right)\left(r + \epsilon^*\left(1\right)\right) - f\left(\epsilon^*\left(1\right)\right) = r\left(R - c\right),$$

which is just an equation that has a unique finite solution, $\epsilon^*\left(1\right)$, because the expression on the left-hand side of it is strictly increasing in ϵ. Therefore, function $\epsilon^*\left(\alpha\right)$ is strictly increasing and bounded.

It means that every time instance, as long as $\alpha > \frac{c}{R}$, the project will be funded.

Second, I show that the efficient funding rate strictly decreases indefinitely over time if $f''\left(x\right) > 0$ for all $x \geqslant 0$ and prior $\alpha_0 > \frac{c}{R}$.

Given (III.a),

$$\dot{\alpha}_t = -\alpha_t\epsilon^*\left(\alpha_t\right)\left(1-\alpha_t\right) < 0,$$

it is obvious that α_t decreases in time as long as $\alpha_t > \frac{c}{R}$. Since $\epsilon^*\left(\alpha_t\right)$ is strictly increasing in α_t, then it is strictly decreasing in time:

$$\dot{\epsilon^*}\left(\alpha_t\right) = \epsilon^{*\prime}\left(\alpha_t\right)\dot{\alpha}_t = -\alpha_t\epsilon^*\left(\alpha_t\right)\left(1-\alpha_t\right)\epsilon^{*\prime}\left(\alpha_t\right) < 0.$$

What is left to do is to show that funds are provided indefinitely if the effort cost function is strictly convex. To do so, I establish a Lemma.

Lemma III.1. *If policy function $\epsilon(\alpha)$ satisfies*

- *integrability on $[\underline{\alpha}, 1]$,*
- *$\epsilon(\alpha) > 0$ for $\alpha \in (\underline{\alpha}, 1]$,*
- *$\epsilon(\underline{\alpha}) = 0$,*
- *and $\epsilon'(\underline{\alpha}) < \infty$,*

then the projects with prior $\alpha_0 > \underline{\alpha}$ are funded indefinitely.

Proof of the Lemma can be found in Appendix III.B. The intuition for why it works is simple. I construct a critical policy function such that if an actual policy function is almost everywhere strictly bellow this critical function, then funding never stops. If it is not, then funding stops in finite time.

Function $\epsilon^*(\alpha)$ is continuous (as a solution to the ODE), bounded, so it is integrable, satisfies $\epsilon^*\left(\frac{c}{R}\right) = 0$ and $\epsilon^*(\alpha) > 0$ for $(\underline{\alpha}, 1]$. It is enough to show that it has a bounded derivative at $\frac{c}{R}$.

Take limits:

$$\lim_{\alpha \to \frac{c}{R}} \epsilon^{*\prime}(\alpha) = \frac{rR^2\left(R - f''(0)\,\epsilon^{*\prime}\left(\frac{c}{R}\right)\right)}{f''(0)\,\epsilon^{*\prime}\left(\frac{c}{R}\right)(R-c)\,c}.$$

If $\epsilon^{*\prime}\left(\frac{c}{R}\right) = \infty$, then this expression is inconsistent, and so it must be the case that $\epsilon^{*\prime}\left(\frac{c}{R}\right) < \infty$ as desired.

Therefore, the efficient funding rate strictly decreases continuously and indefinitely over time.

□

3.3. Discussion

The main result of the analysis of the first best environment is that projects with the prior high enough to get started will be funded either until they are successful or indefinitely at a strictly decreasing rate.

Under the full information environment, the first best outcome is implementable. Since the entrepreneur's actions are observable and verifiable, the contingent contracting is possible. Consider a contract in which the investor provides funds to the entrepreneur each period t according to the efficient policy function $\epsilon^*(\alpha_t)$. If the entrepreneur does not spend everything on experiments and diverts a portion of the funds, then she must pay a crippling fine to the investor. However, if she complies, then she might receive a share of the surplus, s_t, if the project is a success at time t.

To see why the first best outcome is possible in such a scenario, observe that the investor holds full bargaining power, therefore, he will make the entrepreneur break even each period:

$$V(\alpha_t, \alpha_t) = 0,$$

and so

$$\frac{\mathrm{d}V(\alpha, \alpha)}{\mathrm{d}\alpha} = 0.$$

Diverting the funds is a very bad idea as this action is observable, verifiable, and severely punishable, so the entrepreneur always invests the amount she is told to invest. Hence

$$rV(\alpha, \alpha) = \alpha s \epsilon^*(\alpha) R - f(\epsilon^*(\alpha)) + \epsilon^*(\alpha) c - \epsilon^*(\alpha) c = 0$$

and

$$s = \frac{f(\epsilon^*(\alpha))}{\alpha \epsilon^*(\alpha) R}.$$

This is the optimal share. Insert it into the investor's problem and allow him to adjust the funding rate to see if the efficient rate is optimal in this case:

$$rW(\alpha) = \max_{\epsilon} \left[\alpha\epsilon \left(1 - \frac{f(\epsilon)}{\alpha\epsilon R}\right) R - \gamma c - \alpha\epsilon \left(W(\alpha) + (1-\alpha) W'(\alpha)\right) \right],$$

or

$$rW(\alpha) = \max_{\epsilon} \left[\alpha\epsilon R - f(\epsilon) - \gamma c - \alpha\epsilon \left(W(\alpha) + (1-\alpha) W'(\alpha)\right)\right],$$

which is exactly the problem the social planner was solving, so it must have the same solution, the same policy function, $\epsilon^*(\alpha)$.

4. The Equilibrium: Observable but Unverifiable Effort

4.1. Description

In this environment, the entrepreneur's actions are observable by the investor, but it is prohibitively complicated to verify them in the court. It may be due to the nature of the job the entrepreneur is doing: the complexity of experiments may be indescribable to the outsiders, or because despite that both the parties perfectly observe the state, nobody else in the world does, and so the only way to get the information about the entrepreneur's actions is through the investor or the entrepreneur, and both of them may have their own version of the story.

In such an environment, contingent contracting is practically impossible: if the entrepreneur decides to divert the funds, there will be no way to verify that. Thus, the only contingency possible is on the event of success (or alternatively, on no success). This is exactly how offering a share of surplus s_t and transfer γ_t at time t works: $s_t R$ will be paid to the entrepreneur only if the project is a success, while transfer γ_t is intended to cover her costs.

One potential contract may include the clause that if the entrepreneur ever diverts the funds, no funds will be provided in the future ever again, but this threat is not credible as long as there are no better alternatives and as long as there is a potential surplus to extract in the relationship. The investor is stuck with the entrepreneur. Therefore, a better contract may rely on the grim trigger strategy that reverts to the worst possible outcome described here.

Let us consider the entrepreneur's problem. The state is observable, so the beliefs of both players are consistent and there is no hidden information. The problem is fully described and solved in Appendix III.D.

Since $\alpha_t = \hat{\alpha}_t$ for every t, then (III.1), the entrepreneur's problem, is just:

$$rV(\alpha, \alpha) = \max_{\epsilon} \left[\alpha \epsilon s R - f(\epsilon) + \gamma c - \epsilon c - \alpha \epsilon \left(V(\alpha, \hat{\alpha}) + (1-\alpha) \frac{\mathrm{d}V(\alpha, \alpha)}{\mathrm{d}\alpha} \right) \right].$$

The participation constraint will not bind: the entrepreneur can, at least, always divert the funds.

The resulting value function is

$$rV(\alpha, \alpha) = f'(\epsilon^{**}(\alpha)) \epsilon^{**}(\alpha) - f(\epsilon^{**}(\alpha)) + \gamma(\alpha) c \tag{III.7}$$

and so

$$r\frac{\mathrm{d}V\left(\alpha,\alpha\right)}{\mathrm{d}\alpha}=f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)\epsilon^{**\prime}\left(\alpha\right)+\gamma'\left(\alpha\right)c.$$

The investor will always want to be on the entrepreneur's equilibrium path. Suppose not, then if at some time t the investor decides to provide less funds, say

$$\gamma_t<\epsilon^{**}\left(\alpha_t\right),$$

then the entrepreneur will have to spend only $\gamma_t c$ on experimentation. Given the concavity of her objective function, she will want to invest as close to $\epsilon^{**}\left(\alpha\right)c$ as possible. So this is not the best decision on the investor's side: he could have offered a lower share, s_t, and achieved the same outcome (use Figure III.3 as a reference to see why it works, as the decision process is similar enough).

Suppose, he suddenly decided to offer more funds than the entrepreneur is willing to allocate towards the project:

$$\gamma_t>\epsilon^{**}\left(\alpha\right).$$

This decision will not influence $\epsilon^{**}\left(\alpha\right)$, as γ_t does not affect the instantaneous first order condition, and so the entrepreneur will still invest $\epsilon^{**}\left(\alpha\right)c$. The difference

$$\gamma_t c-\epsilon^{**}\left(\alpha\right)c$$

will be diverted. This is a clear waste from the perspectives of the investor.

Therefore, the investor will always want to satisfy

$$\gamma_t=\epsilon^{**}\left(\alpha\right).$$

Consider now the investor's problem. Use (III.2) and the entrepreneur's first order condition as an incentive constraint, keeping in mind that $\gamma=\epsilon$ along the desired path:

$$rW\left(\alpha\right)=\max_{s,\,\epsilon}\left[\alpha\epsilon\left(1-s\right)R-\epsilon c-\alpha\epsilon\left(W\left(\alpha\right)+\left(1-\alpha\right)W'\left(\alpha\right)\right)\right]$$

subject to

$$\alpha sR-f'\left(\epsilon\right)-c-\alpha\left(V\left(\alpha,\hat{\alpha}\right)+\left(1-\alpha\right)\frac{\mathrm{d}V\left(\alpha,\alpha\right)}{\mathrm{d}\alpha}\right)=0,$$

and the participation constraint.

Ignore the participation constraint for now and express the problem in the Lagrangian form:

$$\begin{aligned} rW(\alpha) = \max_{s,\,\epsilon} \Big[& \alpha\epsilon(1-s)R - \epsilon c - \alpha\epsilon\left(W(\alpha) + (1-\alpha)W'(\alpha)\right) \\ & + \lambda\left(\alpha s R - f'(\epsilon) - c - \alpha\left(V(\alpha,\hat{\alpha}) + (1-\alpha)\frac{\mathrm{d}V(\alpha,\alpha)}{\mathrm{d}\alpha}\right)\right)\Big]. \end{aligned}$$

There are two first order conditions: first, with respect to ϵ,

$$\alpha(1-s(\alpha))R - c - \alpha\left(W(\alpha) + (1-\alpha)W'(\alpha)\right) - \lambda f''\left(\epsilon^{**}(\alpha)\right) = 0,$$

and second, with respect to s,

$$-\alpha\epsilon^{**}(\alpha)R + \lambda(\alpha)\alpha R = 0.$$

From the second condition, it is obvious that

$$\lambda(\alpha) = \epsilon^{**}(\alpha),$$

and so it yields

$$rW(\alpha) = f''\left(\epsilon^{**}(\alpha)\right)\left(\epsilon^{**}(\alpha)\right)^2, \tag{III.8}$$

which is positive for all positive ϵ, hence the participation constraint is not binding. Also,

$$rW'(\alpha) = f'''\left(\epsilon^{**}(\alpha)\right)\left(\epsilon^{**}(\alpha)\right)^2\epsilon^{**\prime}(\alpha) + 2f''\left(\epsilon^{**}(\alpha)\right)\epsilon^{**}(\alpha)\,\epsilon^{**\prime}(\alpha).$$

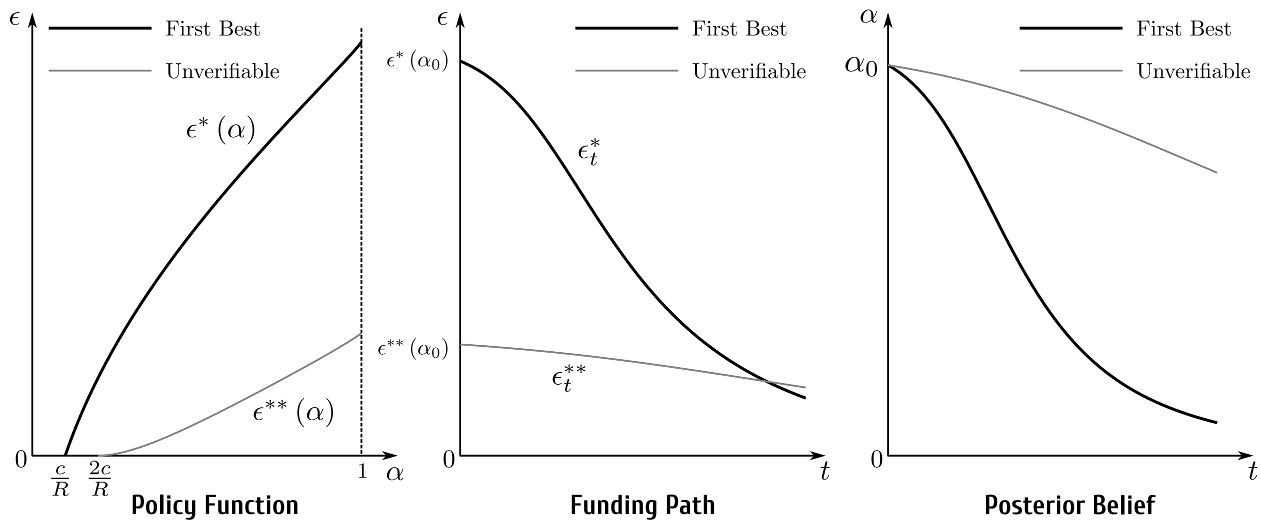
These last two expressions together with the first order conditions are important steps in the solution that make it possible to characterize the equilibrium funding rate in this environment.

4.2. Main Results

The main results of finding the optimal funding rate in the observable but unverifiable effort environment are characterized in Proposition III.2 and shown in Figure.

Proposition III.2. *If the prior belief that the project is good, α_0, is in $\left(\frac{2c}{R}, 1\right)$, and if the effort cost function is strictly convex, then the equilibrium funding rate in the environments with observable but unverifiable effort is always inefficient and strictly decreases over time. Projects with prior less than $\frac{2c}{R}$ are never funded.*

Figure III.5. Observable but Unverifiable Effort Environment



Proof. Intermediate steps of the proof can be found in Appendix III.D. Only the intuition is described here. I first show that the funding rate is everywhere inefficient. Suppose, on the contrary, it were efficient or even higher than efficient, that is, suppose that for some α,

$$\epsilon^{**}(\alpha) \geqslant \epsilon^*(\alpha).$$

Then it must be the case that value function (III.7) together with value function III.8 are higher than the combined social value (III.5) in the case of the first best:

$$\begin{aligned} rV(\alpha,\alpha) + rW(\alpha) &= f''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2 + f'(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) - f(\epsilon^{**}(\alpha)) + \epsilon^{**}(\alpha)c \\ &> f'(\epsilon^*(\alpha))\epsilon^*(\alpha) - f(\epsilon^*(\alpha)) = r\mathcal{V}(\alpha), \end{aligned}$$

which is impossible, hence $\epsilon^{**}(\alpha) < \epsilon^*(\alpha)$ everywhere and funding schedule is inefficient.

Second, I show that function $\epsilon^{**}(\alpha)$ is bounded and it is strictly increasing. Consider the ordinary differential equation

$$\begin{aligned} &r(\alpha R - c) - [f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) + f'(\epsilon^{**}(\alpha)) + c](r + \alpha\epsilon^{**}(\alpha)) + \alpha f(\epsilon^{**}(\alpha)) \\ &= \alpha(1-\alpha)\epsilon^{**\prime}(\alpha)\left[f'''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2 + 3f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) + c\right], \end{aligned}$$

which is the condition that describes the equilibrium funding path in the case of observable but unverifiable effort and produces $\epsilon^{**}(\alpha)$ when solved, given some particular function $f(\cdot)$ with $f(0) = 0$ and $f'(0) = 0$.

The lowest posterior belief is $\frac{2c}{R}$—twice as high as the lowest posterior in the first best case. Given an equal share of $\frac{1}{2}$, this belief level makes the investor break even:

$$\alpha\epsilon(1-s)R - \epsilon c = \frac{2c}{R}\epsilon\frac{1}{2}R - \epsilon c = \epsilon c - \epsilon c 0,$$

while making the entrepreneur stop experimenting:

$$\alpha\epsilon s R - f(\epsilon) - \epsilon c = \frac{2c}{R}\epsilon\frac{1}{2}R - f(\epsilon) - \epsilon c = \epsilon c - f(\epsilon) - \epsilon c = -f(\epsilon),$$

so $\epsilon^{**}\left(\frac{2c}{R}\right) = 0$.

Insert $\epsilon^{**}(\alpha) = 0$ to the ODE to find the desired lowest possible $\underline{\alpha}$, which still makes it possible to agree on continuing with the project:

$$r(\underline{\alpha}R - 2c) = \underline{\alpha}(1-\underline{\alpha})\epsilon^{**\prime}(\underline{\alpha})c.$$

The left-hand side of this expression is positive only if $\underline{\alpha} \geqslant \frac{2c}{R}$, the right-hand side is positive always when funding rate increases. Therefore, there is always a room for agreement as long as $\alpha \geqslant \frac{2c}{R}$, which implies $\underline{\alpha} = \frac{2c}{R}$. This is the lower bound.

To get the upper bound, consider $\alpha = 1$. Then the ODE becomes

$$r(R-c) = \left[f''(\epsilon^{**}(1))\epsilon^{**}(1) + f'(\epsilon^{**}(1)) + c\right](r + \epsilon^{**}(1)) - f(\epsilon^{**}(1)),$$

and it has has a unique solution $\epsilon^{**}(1)$, which is an upper bound.

Therefore, function $\epsilon^{**}(\alpha)$ is bounded if it increases everywhere between $\frac{2c}{R}$ and 1.

To see why function $\epsilon^{**}(\alpha)$ actually strictly increases on $\left(\frac{2c}{R}, 1\right)$, suppose that it does not. Then there must be extreme points of function $\epsilon^{**}(\alpha)$ somewhere on $\left(\frac{2c}{R}, 1\right)$. Differentiate the ODE with respect to α and assume $\epsilon^{**\prime}(\alpha) = 0$, then assume the same for the ODE itself and combine the two to produce

$$\frac{r}{\alpha}\left[(1-\alpha)\left(f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) + f'(\epsilon^{**}(\alpha)) + c\right) + c\right]$$

$$= \alpha (1-\alpha) \epsilon^{**\prime\prime}(\alpha) \left[f'''(\epsilon^{**}(\alpha)) (\epsilon^{**}(\alpha))^2 + 3f''(\epsilon^{**}(\alpha)) \epsilon^{**}(\alpha) + c \right].$$

The left-hand side of the expression is strictly positive, so the right-hand side must be positive as well, so $\epsilon^{**\prime\prime}(\alpha) > 0$, which implies that all the extreme points that can be found for function $\epsilon^{**}(\alpha)$ are local minima. This is impossible: either there is one local minimum without any local maxima, but then it means that the function is negative somewhere; or there must be at least one local maximum due to the smoothness of the function (as a solution to the ODE), which contradicts the finding that all the extreme points are local minima.

Therefore, function $\epsilon^{**}(\alpha)$ strictly increases everywhere on the interior of the range. So

$$\epsilon^{**\prime}(\alpha) > 0$$

and

$$\frac{\mathrm{d}\epsilon^{**}(\alpha_t)}{\mathrm{d}t} = -\alpha_t \epsilon^{**}(\alpha_t)(1-\alpha_t) \epsilon^{**\prime}(\alpha_t) < 0.$$

Thus, the funding rate strictly decreases in time for all prior $\alpha_0 \in \left(\frac{2c}{R}, 1\right)$.

The only thing left to do is to show that funding never stops. Function $\epsilon^{**}(\alpha_t)$ is bounded and continuous, so it is integrable on $\left[\frac{2c}{R}, 1\right]$; it is strictly positive for all $\alpha \in \left(\frac{2c}{R}, 1\right]$; it is true that $\epsilon^{**}\left(\frac{2c}{R}\right) = 0$; and finally $\epsilon^{**\prime}\left(\frac{2c}{R}\right) = 0 < \infty$.

Hence, all the conditions of Lemma III.1 are satisfied and funding never stops.

Therefore, if the prior is less than one, but higher than $\frac{2c}{R}$, the funding rate in the observable but unverifiable action environment is always inefficient and it decreases indefinitely it time conditional on no success. □

4.3. Discussion

This Markovian solution does not employ the properties of the infinite time that both the players may use to their advantage: a Folk theorem could possibly be used to come up with a better outcome. However, the characterized solution can be used as the lowest equilibrium benchmark to which the players can revert if they fail to implement a better solution. The actual property and characterization of the strategies that implement a better outcome is beyond the purpose of this chapter as it relies on non-Markovian equilibria concepts. However, the intuition is simple: the investor will promise a better share and provide more funds to the entrepreneur as long as he sees that no deviation happened in the past. If the entrepreneur deviates, then the play immediately reverts to the Markov equilibrium described above.

5. The Equilibrium: Unobservable Effort

5.1. Description

Finally, let us consider the environment in which the entrepreneur's actions are completely unobservable by the investor. In this setup, providing incentives to the principal to stay on the equilibrium funding path will be of grave importance for the investor. Appendix III.E is devoted to the development of the problem and to the characterization of the equilibrium conditions.

In this environment, the only contingency possible in contracts is the contingency on the event of success or the lack thereoff. Share s_t offered by the investor each period exploits this possibility as $s_t R$ is paid to the entrepreneur only if the project succeeds.

Start with the entrepreneur's problem. She received the offer, accepted it, and needs to decide upon her effort level. She believes that the probability that the project is good is α and she knows that the investor believes that this probability is $\hat{\alpha}$. She solves

$$\begin{aligned} rV(\alpha, \hat{\alpha}) = \max_{\epsilon} \big[& (\alpha \epsilon s R - f(\epsilon) - \epsilon c + \gamma c) - \alpha \epsilon \left(V_1(\alpha, \hat{\alpha})(1-\alpha) + V(\alpha, \hat{\alpha}) \right) \\ & - V_2(\alpha, \hat{\alpha}) \hat{\alpha} \hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha}) \big]. \end{aligned}$$

The corresponding first order condition is

$$\alpha s R - f'(\epsilon^{**}) - c - \alpha \left[V_1(\alpha, \hat{\alpha})(1-\alpha) + V(\alpha, \hat{\alpha}) \right] = 0.$$

The second order condition is satisfied automatically.

Now that the investor knows that the entrepreneur's best response depends on the investor's offer schedule, he can solve his problem having the entrepreneur's first order condition as a constraint:

$$rW(\hat{\alpha}) = \max_{s, \hat{\epsilon}} \left[\hat{\alpha} \hat{\epsilon}(1-s) R - \hat{\epsilon} c - \hat{\alpha} \hat{\epsilon} \left(W(\hat{\alpha}) + (1-\hat{\alpha}) W'(\hat{\alpha}) \right) \right]$$

subject to

$$\hat{\alpha} s R - f'(\hat{\epsilon}) - c - \alpha \left[V_1(\hat{\alpha}, \hat{\alpha})(1-\hat{\alpha}) + V(\hat{\alpha}, \hat{\alpha})\right] = 0,$$

and the participation constraint that I will ignore for now.

In the Lagrangian form, the problem becomes

$$rW(\hat{\alpha}) = \max_{s,\hat{\epsilon}} \left[\hat{\alpha}\hat{\epsilon}(1-s) R - \hat{\epsilon} c - \hat{\alpha}\hat{\epsilon}\left(W(\hat{\alpha}) + (1-\hat{\alpha}) W'(\hat{\alpha})\right)\right.$$
$$\left. + \lambda\left(\hat{\alpha} s R - f'(\hat{\epsilon}) - c - \alpha\left[V_1(\hat{\alpha}, \hat{\alpha})(1-\hat{\alpha}) + V(\hat{\alpha}, \hat{\alpha})\right]\right)\right]$$

There are two first order conditions. The first one, with respect to $\hat{\epsilon}$:

$$\hat{\alpha}(1 - s(\hat{\alpha})) R - c - \hat{\alpha}\left(W(\hat{\alpha}) + (1-\hat{\alpha}) W'(\hat{\alpha})\right) - \lambda f''(\hat{\epsilon}^{**}(\hat{\alpha})) = 0,$$

and the second one, with respect to s:

$$-\hat{\alpha}\hat{\epsilon}^{**}(\hat{\alpha}) R + \lambda(\hat{\alpha}) \hat{\alpha} R = 0.$$

From the second condition

$$\lambda(\hat{\alpha}) = \hat{\epsilon}^{**}(\hat{\alpha}).$$

Hence

$$rW(\hat{\alpha}) = f''(\hat{\epsilon}^{**}(\hat{\alpha}))\left(\hat{\epsilon}^{**}(\hat{\alpha})\right)^2,$$

which is positive for positive funding rates, hence no need for the explicit participation constraint, and

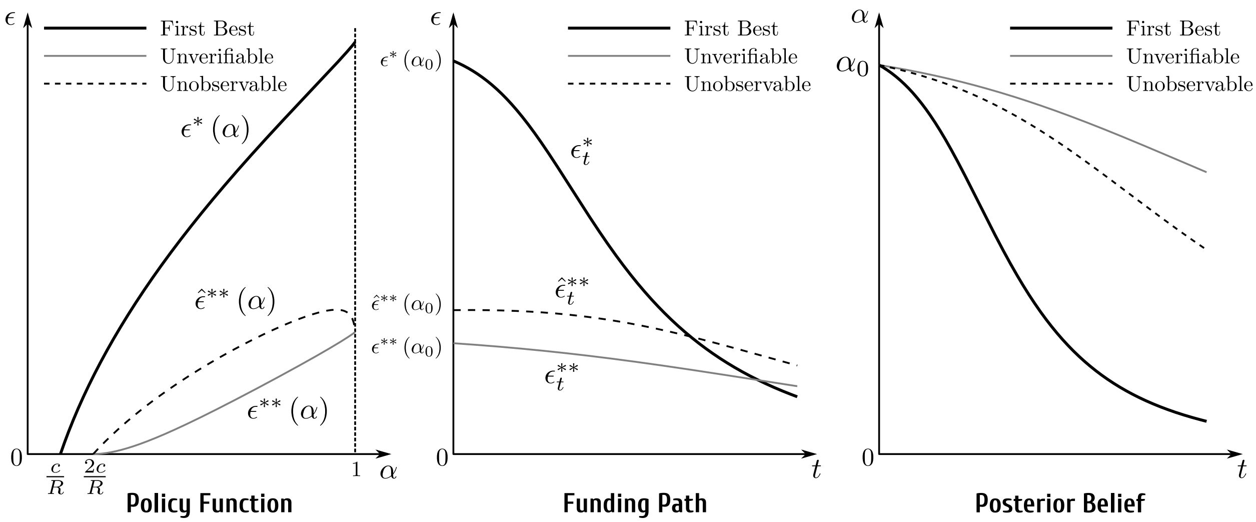
$$rW'(\hat{\alpha}) = f'''(\hat{\epsilon}^{**}(\hat{\alpha}))\left(\hat{\epsilon}^{**}(\hat{\alpha})\right)^2 \hat{\epsilon}^{**\prime}(\hat{\alpha}) + 2 f''(\hat{\epsilon}^{**}(\hat{\alpha})) \hat{\epsilon}^{**}(\hat{\alpha}) \hat{\epsilon}^{**\prime}(\hat{\alpha}).$$

So the investor's value function increases in the state variable if the funding schedule increases in the state variable.

5.2. Main Results

I establish the main results for the unobservable effort environment in Proposition III.3 and demonstrate in Figure III.6. This environment is more complicated than the first-best and the observable but unverifiable effort environments. However, it is possible to establish that the funding rate will *eventually* be strictly decreasing over time. At the beginning, however, it is possible that the project will be funded at an increasing rate.

Figure III.6. Unobservable Effort Environment



Proposition III.3. *If the prior belief that the project is good, α_0, is in $\left(\frac{2c}{R}, 1\right)$ and if the effort cost function is strictly convex, then the equilibrium funding rate in the environments with unobservable effort eventually becomes strictly decreasing in time and then funding strictly decreases until the project is successful or indefinitely. Projects with the prior lower than $\frac{2c}{R}$ are never funded.*

Proof. Some important steps of this proof can be found in Appendix III.E.

I begin by showing that the policy function, $\hat{\epsilon}^{**}$, is bounded. The lower bound is the same as in the environment with the observable but unverifiable effort. The intuition is simple: as long as there exist a surplus to extract and a share that allows the investor and the entrepreneur to consider experimentation, then there will be a possibility to write contracts. When $\underline{\alpha} = \frac{2c}{R}$, the optimal share is $s = \frac{1}{2}$, and the level of investment is zero. This is the lower bound.

Finding the upper bound is more complicated in this environment, but it exists. The second-order ordinary differential equation that yields function $\hat{\epsilon}^{**}(\hat{\alpha})$ when solved, given some function $f(\cdot)$, degenerates to

$$\begin{aligned} &r\left(R - f'\left(\hat{\epsilon}^{**}(1)\right) - f''\left(\hat{\epsilon}^{**}(1)\right)\hat{\epsilon}^{**}(1) - 2c\right) \\ &- \left[f'\left(\hat{\epsilon}^{**}(1)\right)\hat{\epsilon}^{**}(1) - f\left(\hat{\epsilon}^{**}(1)\right) + f''\left(\hat{\epsilon}^{**}(1)\right)\left(\hat{\epsilon}^{**}(1)\right)^2 + \hat{\epsilon}^{**}(1)\,c\right] = 0, \end{aligned}$$

when $\hat{\alpha} = 1$. This is the same result as in the observable but unverifiable case, so $\hat{\epsilon}^{**}(1) = \epsilon^{**}(1)$. Thus, function $\hat{\epsilon}^{**}(\hat{\alpha})$ connects two points with finite values, it has finite first derivatives between these two points, and it is a solution to the ordinary differential equation, so it must be bounded above as well. If it strictly increases in $\hat{\alpha}$, then the upper bound is obvious, $\hat{\epsilon}^{**}(1)$, otherwise, it is located somewhere on the interior of the range.

Second, I show that $\hat{\epsilon}^{**\prime}(\underline{\alpha}) < \infty$. Divide both sides of the differential equation that produces the policy function by $\hat{\epsilon}^{**}(\hat{\alpha})$ and take limits as $\hat{\alpha} \to \frac{2c}{R}$:

$$r\left(\frac{R}{\hat{\epsilon}^{**\prime}\left(\frac{2c}{R}\right)} - 2f''(0)\right) + \frac{R-4c}{R}c$$
$$= 8\frac{c(R-2c)}{R^2}f''(0)\,\hat{\epsilon}^{**\prime}\left(\frac{2c}{R}\right) + 2\frac{4c^2(R-2c)^2}{rR^4}f''(0)\left(\hat{\epsilon}^{**\prime}\left(\frac{2c}{R}\right)\right)^2.$$

It is clear that $\hat{\epsilon}^{**\prime}\left(\frac{2c}{R}\right) = \infty$ is incompatible with this condition, so it must be the case that $\hat{\epsilon}^{**\prime}\left(\frac{2c}{R}\right) < \infty$.

Third, I show that the project will be funded indefinitely. Function $\hat{\epsilon}^{**}$ is bounded and continuous as a solution to the second-order ordinal differential equation, so it is integrable on $\left[\frac{2c}{R}, 1\right]$. Funding is positive everywhere on the interior of this interval as the project is viable and there is a room for agreement. Also, $\hat{\epsilon}^{**}\left(\frac{2c}{R}\right) = 0$ and $\hat{\epsilon}^{**\prime}\left(\frac{2c}{R}\right) < \infty$. Therefore, all the requirements for Lemma III.1 are satisfied, and so the project is funded indefinitely or until success is achieved.

Given continuity and the fact that $\hat{\epsilon}^{**}\left(\frac{2c}{R}\right) = 0$, it must be the case that there exist some belief $\hat{\alpha}$‘, such that for all $\hat{\alpha} \leqslant \hat{\alpha}$‘, $\hat{\epsilon}^{**\prime}(\hat{\alpha}) > 0$. It means that when the belief level of $\hat{\alpha}$‘ is reached, the funding rate begins to decrease in time indefinitely. □

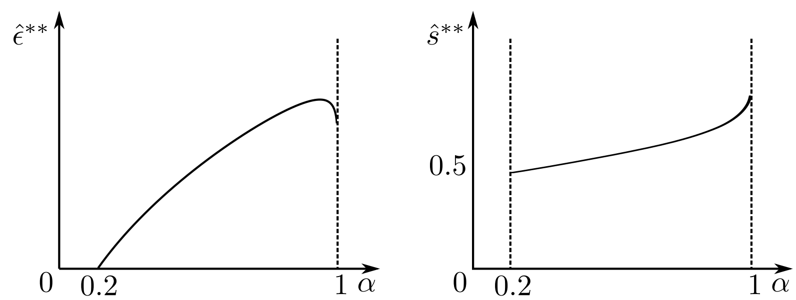
5.3. Discussion

The fact that the funding rate may be increasing can be explained if we consider the entrepreneur's incentives:

$$\alpha s R - \alpha\left[V_1(\alpha, \hat{\alpha})(1-\alpha) + V(\alpha, \hat{\alpha})\right] = f'(\epsilon^{**}) + c.$$

Consider an example, in which $f(x) = x^2$, $R = 20$, $c = 2$, and $r = 0.05$. The equilibrium funding rate and the share schedules for this example are demonstrated in Figure III.7. Notice that for α

Figure III.7. An Example of the Policy Function with a Decreasing Region



close to one, the shares, which are offered to the entrepreneur, increase with α (decrease with time). Hence as the time passes, αsR decreases. In such circumstances, for the funding rate to increase in time, it must be the case that the opportunity cost $\alpha\left[V_1\left(\alpha,\,\hat{\alpha}\right)(1-\alpha)+V\left(\alpha,\,\hat{\alpha}\right)\right]$ decreases fast, which is exactly what happens when the entrepreneur becomes more pessimistic given high initial level of optimism. The investor exploits this decrease in the opportunity cost to his advantage: now he can offer higher funding rates for a lower level of optimism, α.

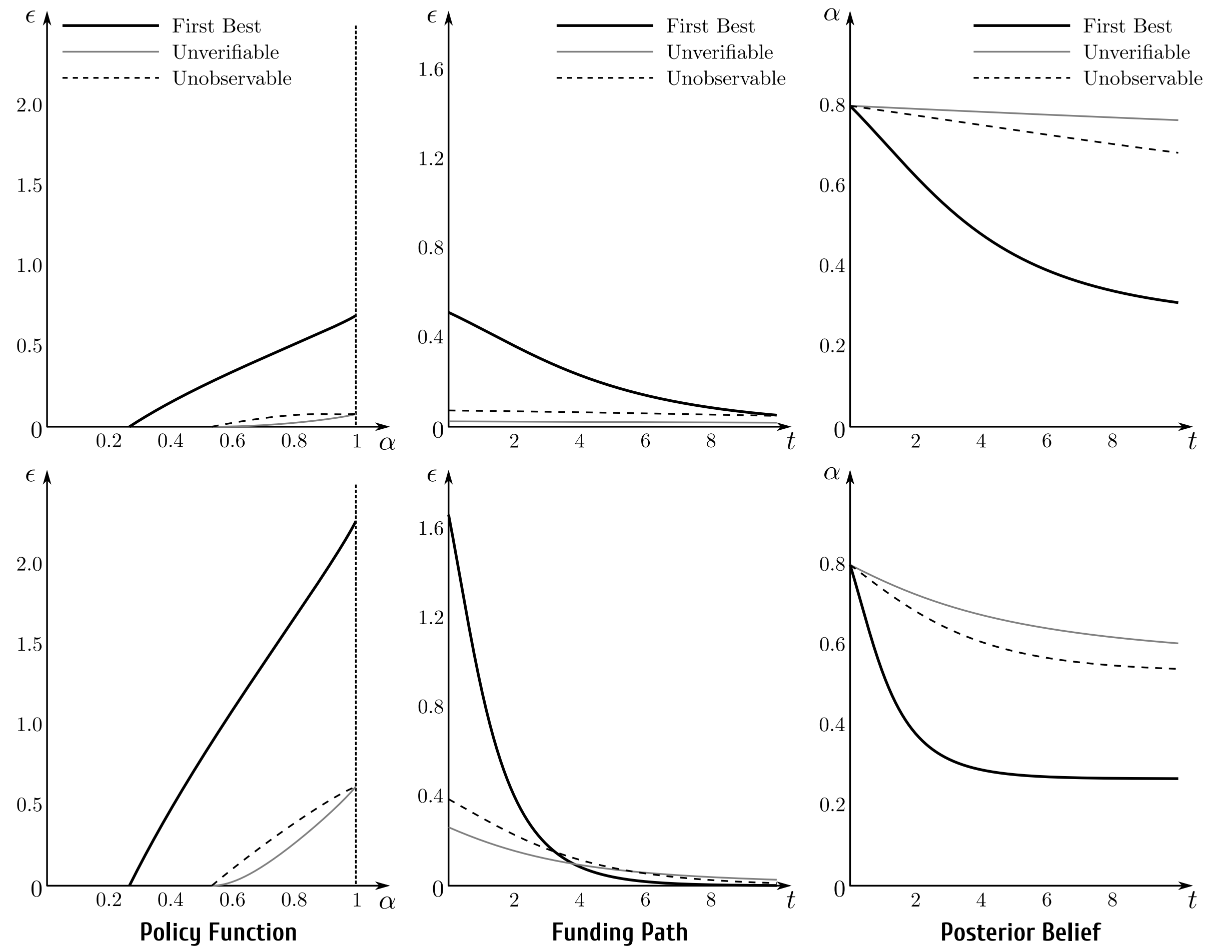
6. Comparative Statics

6.1. Change of Parameters

It is interesting to understand how the funding paths depend of the parameters of the model. The main parameters of the model are the discount rate, r, the total surplus of size R, and the marginal monetary cost of effort, c. I consider four different cases:

1. Patient players (r is low), high surplus ($\frac{R}{c}$ is high);
2. Impatient players (r is high), high surplus;
3. Patient players, low surplus ($\frac{R}{c}$ is low);
4. Impatient players, low surplus.

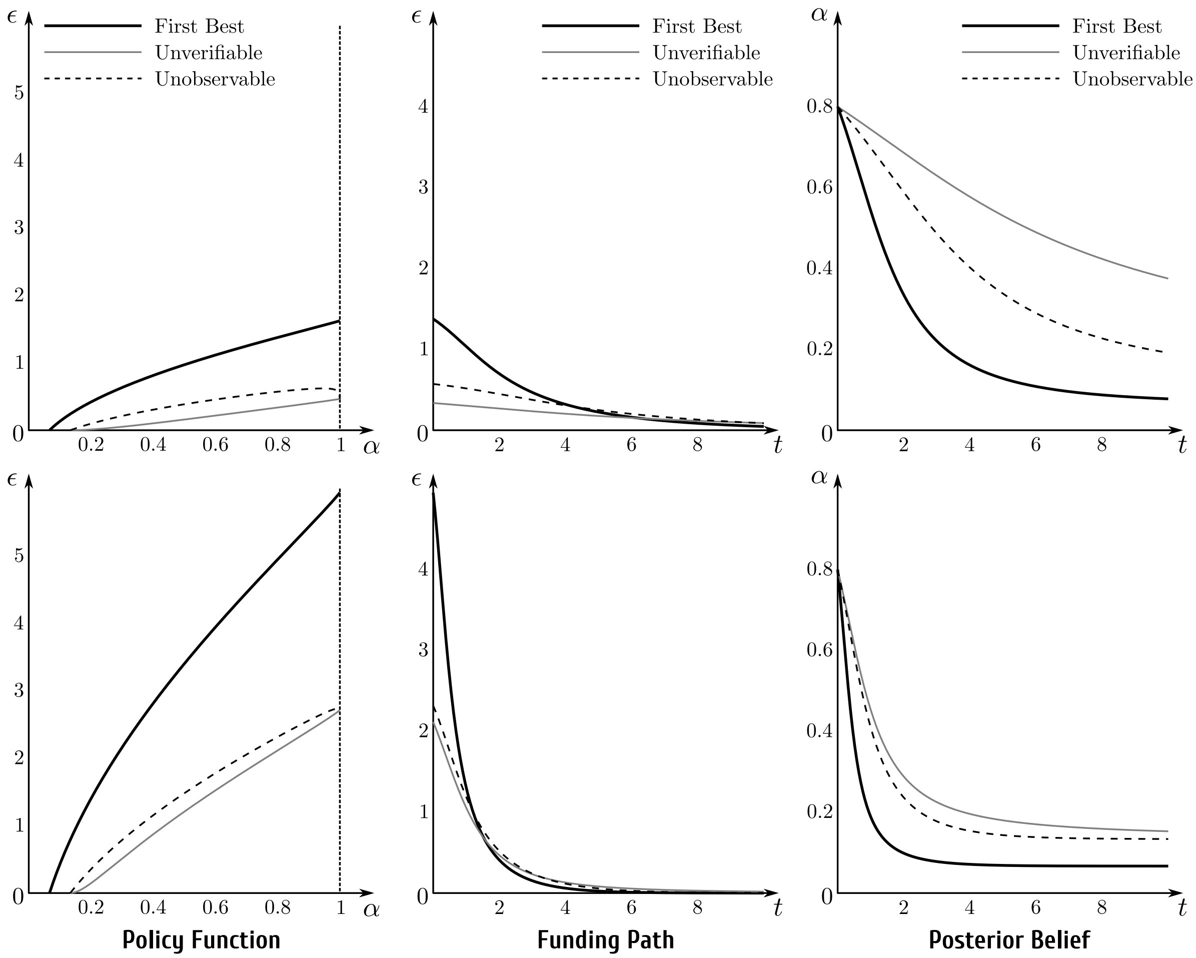
Figure III.8. Comparative Statics: Low Surplus, Patience (Top) and Impatience (Bottom)



These four cases are presented in Figures III.8 and III.9. The left-hand sides of the figures demonstrate how funding rates depend on the level of optimism. The middle parts depict the funding rates. The right-hand sides show how the posterior beliefs evolve over time. The top portions of the figures show the "patient" cases, while the bottom parts—the "impatient." Notice that everywhere the "unobserved" policy function is higher than the "observable but unverifiable" policy function.

The patience measure significantly affects the funding path. If the players are impatient, they begin experimenting at the high funding rate, which quickly decreases over time. The most profound this effect is for the first-best funding rate. Since all the funding rates are higher for the

Figure III.9. Comparative Statics: High Surplus, Patience (Top) and Impatience (Bottom)



impatient players, it means that impatience allows completing the projects faster. Being patient means relying more on the potential success in the future.

The second component that affects the funding rates tremendously is the profitability rate, $\frac{R}{c}$, which is high for the high-surplus projects and low for the low-surplus projects. A decrease in the surplus size shifts policy functions to the right, which means that the projects now require much higher level of optimism to be worked on. The decrease in the surplus size also changes the slope of the functions making them flatter, which means that the funding rates become lower everywhere. This effect is especially pronounced for the equilibria environments: having a low surplus means that the funding rates become vanishingly small.

The high-surplus projects that have a prior close to one demonstrate funding rates that will increase in time at the beginning, and will become strictly decreasing as the time goes on. This effect, however, is not significantly pronounced in these examples. More research is needed to determine the exact conditions when the funding rates may be increasing at the beginning.

6.2. Time

The traditional implication that projects are abandoned too early should be taken carefully. What it essentially means is that projects are underfunded, not that the time spent on projects is sub-optimal in general.

Notice that in the presence of the convex effort costs, it takes longer to reach a certain level of pessimism as compared to the first best. In the end, it actually looks like entrepreneurs and investors spend too much time working on projects while applying too little effort.

6.3. Ignorance is Bliss

An interesting outcome of the numerical analysis is that the policy function in the environment with unobservable effort tends to be everywhere higher (for every α) than the policy function in the environment with the observable but unverifiable effort. The explanation for this is simple: if the entrepreneur diverts, but the investor does not observe the deviation, then the entrepreneur becomes more optimistic and will be willing to invest more than the investor supplies. It is not the case in the situation with the observable but unverifiable actions: the entrepreneur can divert the funds and later effectively force the investor to pay more because the investor will clearly observe that the state has not changed as much as he anticipated, and so he will be more optimistic, and more ready to invest, as well.

In this sense, the ignorance of the state is the device that allows the investor to commit to providing funds at a certain rate without paying attention to the signals that come from the entrepreneur. Surprisingly, the investor will prefer to be uninformed. This will make some threats by the entrepreneur non-credible: even if she actually diverts, the investor will not believe her. This outcome has been discussed in the literature before (see, for example Bergemann and Hege, 2005).

7. Possible Extensions

7.1. Venture Capital Markets

This chapter, as well as many previous papers, traditionally deals with bilateral funding schemes, with a single investor and a single entrepreneur. If there is a continuity of entrepreneurs with different projects, and a continuity of investors with different attitude towards risk, then there are many questions a model with convex costs could answer.

The questions to ask are: which projects will get funded and how the funding schedule will unroll; will projects be abandoned earlier in such a setting; is the first-best result possible if the outside options are high; will the problem of observable but unverifiable environment be solved by negatively screening slackers, and many other interesting questions.

7.2. Hazard Rates

Another model to consider is the model in which the current probability of success of the project depends on the accumulated effort. Thus, there will be two states: the probability that the project is still good, and the accumulated effort. In this environment, Markov strategies will be richer, and the range of potential funding functions wider.

In particular, it will be interesting to find if there are projects that nobody wants to undertake because they require too much initial accumulated effort to become promising. Discovering the hazard rate functions that produce the funding rates which are locally increasing in time and characterizing the set of such functions would be a good starting point.

8. Conclusions

I develop a solution technique and characterize the Markov sequential equilibria of the dynamic agency model with convex effort costs and three information environments. I demonstrate that the funding rates eventually become strictly decreasing in time. I show that patience and surplus size play important roles in how the shape of the funding function looks like and I characterize some promising extensions to the model.

Common Appendices

A. Probability of No Success

Define time T to be the time when the project succeeds. If it never succeeds, then $T = \infty$. Begin by defining the probability of reaching no success by time t:

$$\mathbb{P}(T > t) = 1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}.$$

Then the probability that the project is still good at time t conditional on no success so far is

$$\alpha_t \equiv \mathbb{P}(\text{project is good} \mid T > t) = \frac{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}.$$

The time-derivative of probability α_t is

$$\begin{aligned}\frac{\mathrm{d}\alpha_t}{\mathrm{d}t} \equiv \dot{\alpha}_t &= -\epsilon_t \left[\frac{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}} - \left(\frac{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}\right)^2\right] \\ &= -\alpha_t \epsilon_t (1 - \alpha_t),\end{aligned}$$

therefore,

$$\frac{\mathrm{d} \ln \alpha_t}{\mathrm{d}t} = \dot{\ln \alpha_t} = \frac{\dot{\alpha}_t}{\alpha_t} = -\epsilon_t (1 - \alpha_t).$$

Thus

$$\begin{aligned}\ln \alpha_t &= \ln \alpha_0 + \int_0^t \dot{\ln \alpha_\tau} \, \mathrm{d}\tau \\ &= \ln \alpha_0 - \int_0^t \epsilon_\tau (1 - \alpha_\tau) \, \mathrm{d}\tau.\end{aligned}$$

Now, go back to the probability of no success by time t and produce:

$$\mathbb{P}(T > t) = 1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}$$

$$
\begin{aligned}
&= \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} \frac{1-\alpha_0+\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}} \\
&= \frac{\alpha_0}{\alpha_t} e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} \\
&= \frac{\alpha_0}{e^{\ln \alpha_t}} e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} \\
&= \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau - \ln \alpha_t} \\
&= \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau - \ln \alpha_0 + \int_0^t \epsilon_\tau (1-\alpha_\tau) \, \mathrm{d}\tau} \\
&= \frac{\alpha_0}{\alpha_0} e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau + \int_0^t \epsilon_\tau \, \mathrm{d}\tau - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \\
&= e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}.
\end{aligned}
$$

Therefore,

$$
\mathbb{P}(T > t) = 1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} = e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}.
$$

B. Payoffs and Expectations

Begin by formulating the probability distribution of the random time the project succeeds. Given that

$$
\mathbb{P}(T > t) = e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau},
$$

the cumulative distribution function of random variable T, the time when the project succeeds, is

$$
F(t) = \mathbb{P}(T < t) = 1 - e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}.
$$

Thus the probability density function of random variable T is

$$
p(t) = \alpha_t \epsilon_t e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau}.
$$

Take some general payoff function of the form

$$
\pi = \mathbb{E}\left[e^{-rT} v_T - \int_0^T e^{-rt} k_t \, \mathrm{d}t \right].
$$

Using the expression for the probability density function derived above,

$$
\begin{aligned}
\pi &= \mathbb{E}\left[e^{-rT}v_T - \int_0^T e^{-rt}k_t \, \mathrm{d}t\right] \\
&= \mathbb{E}\left[e^{-rT}v_T\right] - \mathbb{E}\left[\int_0^T e^{-rt}k_t \, \mathrm{d}t\right] \\
&= \int_0^\infty e^{-rT}v_T p\left(T\right) \, \mathrm{d}T - \int_0^\infty \int_0^T e^{-rt}k_t p\left(T\right) \, \mathrm{d}t\mathrm{d}T.
\end{aligned}
$$

Integrate the second term by parts:

$$
\begin{aligned}
&\int_0^\infty \int_0^T e^{-rt}k_t p\left(T\right) \, \mathrm{d}t\mathrm{d}T \\
&= \int_0^T e^{-rt}k_t F\left(T\right) \, \mathrm{d}t\Big|_0^\infty - \int_0^\infty e^{-rT}k_T F\left(T\right) \, \mathrm{d}T \\
&= \int_0^\infty e^{-rt}k_t \, \mathrm{d}t - \int_0^\infty e^{-rt}k_t F\left(t\right) \, \mathrm{d}t \\
&= \int_0^\infty e^{-rt}k_t \left(1 - F\left(t\right)\right) \, \mathrm{d}t \\
&= \int_0^\infty e^{-rt}k_t e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \, \mathrm{d}t \\
&= \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} k_t \, \mathrm{d}t.
\end{aligned}
$$

Thus the general payoff function becomes

$$
\begin{aligned}
\pi &= \mathbb{E}\left[e^{-rT}v_T - \int_0^T e^{-rt}k_t \, \mathrm{d}t\right] \\
&= \int_0^\infty e^{-rT}v_T p\left(T\right) \, \mathrm{d}T - \int_0^\infty \int_0^T e^{-rt}k_t p\left(T\right) \, \mathrm{d}t\mathrm{d}T \\
&= \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \alpha_t \epsilon_t v_t \, \mathrm{d}t - \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} k_t \, \mathrm{d}t \\
&= \int_0^\infty e^{-rt-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left[\alpha_t \epsilon_t v_t - k_t\right] \, \mathrm{d}t.
\end{aligned}
$$

This expression can be used to derive the particular payoff functions of the players.

C. Evolution of Beliefs and Critical Experimentation Rate

This appendix is devoted to the development of the critical experimentation rate that is used to determine if experimentation ever stops. The idea behind this is that if the actual experimentation rate is everywhere below the critical experimentation rate then the experiments are conducted for infinitely many periods of time. I want to know if $\alpha_t > \underline{\alpha}$ approaches lower bound $\underline{\alpha}$ asymptotically as time advances or in finite time. That is, I would like to know if for some t, and for all really small $\Delta > 0$,

$$\left|\dot{\alpha}_t \Delta\right| \leqslant \left|\alpha_t - \underline{\alpha}\right|,$$

or since $\alpha_t \geqslant \underline{\alpha}$ everywhere where we are interested in it to be, and $\dot{\alpha}_t \leqslant 0$,

$$\dot{\alpha}_t \Delta \geqslant \underline{\alpha} - \alpha_t.$$

Substitute the expression for $\dot{\alpha}_t$:

$$\alpha_t \epsilon_t \left(1 - \alpha_t\right) \Delta \leqslant \alpha_t - \underline{\alpha},$$

or

$$\epsilon_t \leqslant \frac{\alpha_t - \underline{\alpha}}{\alpha_t \left(1 - \alpha_t\right) \Delta}.$$

Define

$$\bar{\epsilon}\left(\alpha\right) \equiv \frac{\alpha - \underline{\alpha}}{\alpha \left(1 - \alpha\right) \Delta}.$$

If some experimentation policy function $\epsilon\left(\alpha\right)$ is everywhere below function $\bar{\epsilon}\left(\alpha\right)$, then α_t approaches $\underline{\alpha}$ asymptotically in time, that is, the experimentation never completely stops.

Notice that

$$\bar{\epsilon}\left(\underline{\alpha}\right) = 0,$$

and

$$\lim_{\alpha \to 1} \bar{\epsilon}\left(\alpha\right) = \infty,$$

and everywhere in-between, this function can be made as large as we please by simply changing Δ.

Therefore, if policy function $\epsilon\left(\alpha\right)$ is bounded and continuous, then it is below function $\bar{\epsilon}\left(\alpha_t\right)$ everywhere on $\left(\underline{\alpha}, 1\right]$. The only problematic point is $\alpha = \underline{\alpha}$.

The first derivative of the critical function at $\underline{\alpha}$ is

$$\bar{\epsilon}'\left(\underline{\alpha}\right) = \frac{1}{\underline{\alpha}\left(1 - \underline{\alpha}\right)\Delta} < \infty.$$

It can be made as large as we please by changing Δ, but it will still remain finite. So if a policy function has a vertical tangent at $\underline{\alpha}$, then it cannot be everywhere below $\bar{\epsilon}\left(\alpha\right)$. Hence, finite derivative of a policy function at $\underline{\alpha}$ is crucial.

Finally, it must be the case that $\epsilon\left(\underline{\alpha}\right) = 0$ and $\epsilon\left(\alpha\right) > 0$ so that experimentation never stops in-between.

Chapter I

Appendices

I.A. The Generalized Control Problem

I present and solve the problem of finding the optimal experimentation path in the case when the entrepreneur needs to reach a certain belief level by time T_2 from time T_1.

I begin by formulating the problem from the perspectives of Pontryagin's Maximum Principle:

$$J = \max_{(\epsilon_t, t \in [T_1, T_2])} \left[\int_{T_1}^{T_2} e^{-rt - \int_{T_1}^{t} \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left(\alpha_t \epsilon_t s R - f\left(\epsilon_t\right) - \epsilon_t c \right) \mathrm{d}t \right]$$

$$\text{subject to } \dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right)$$

$$\text{and } \alpha_0, \alpha_T \text{ given.}$$

Observe that

$$M_t \equiv e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} = e^{-\int_0^{T_1} \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} e^{-\int_{T_1}^{t} \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} = M_{T_1} e^{-\int_{T_1}^{t} \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau},$$

and so I can define

$$\bar{M}_t \equiv e^{-\int_{T_1}^{t} \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} = \frac{M_t}{M_{T_1}},$$

$$\dot{\bar{M}} = \frac{\dot{M}_t}{M_{T_1}} = -\alpha_t \epsilon_t \frac{M_t}{M_{T_1}} = -\alpha_t \epsilon_t \bar{M}_t..$$

Rewrite the problem:

$$
\begin{aligned}
J = \max_{(\epsilon_t, t\in[T_1,T_2])} & \left[\int_{T_1}^{T_2} e^{-rt}\bar{M}_t\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right)\,\mathrm{d}t\right] \\
\text{subject to } & \dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right), \\
& \dot{\bar{M}}_t = -\alpha_t\epsilon_t\bar{M}_t \\
\text{and } & \alpha_{T_1}, \alpha_{T_2} \text{ given,} \\
& \bar{M}_{T_1} = 1.
\end{aligned}
\tag{I.a}
$$

Assign Lagrange multipliers λ_t and μ_t to the first and the second constraints, respectively, and rewrite the problem in the Lagrangian form:

$$
\mathcal{L} = \int_{T_1}^{T_2}\left[e^{-rt}\bar{M}_t\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right) + \lambda_t\left(-\alpha_t\epsilon_t\left(1-\alpha_t\right) - \dot{\alpha}_t\right) + \mu_t\left(-\alpha_t\epsilon_t\bar{M}_t - \dot{\bar{M}}_t\right)\right]\,\mathrm{d}t
$$

$$
\begin{aligned}
& \alpha_{T_1}, \alpha_{T_2} \text{ given,} \\
& \bar{M}_{T_1} = 1.
\end{aligned}
$$

Integrate by parts $\lambda_t\dot{\alpha}_t$ and $\mu_t\dot{\bar{M}}_t$ and rewrite the problem:

$$
\begin{aligned}
\mathcal{L} = \int_{T_1}^{T_2} & \left[e^{-rt}\bar{M}_t\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right) - \lambda_t\alpha_t\epsilon_t\left(1-\alpha_t\right) - \mu_t\alpha_t\epsilon_t\bar{M}_t + \dot{\lambda}_t\alpha_t + \dot{\mu}_t\bar{M}_t\right]\,\mathrm{d}t \\
& - \left[\lambda_{T_2}\alpha_{T_2} - \lambda_{T_1}\alpha_{T_1}\right] - \left[\mu_{T_2}\bar{M}_{T_2} - \mu_{T_1}\right].
\end{aligned}
$$

Define Hamiltonian function

$$
\mathcal{H}_t = e^{-rt}\bar{M}_t\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right) - \lambda_t\alpha_t\epsilon_t\left(1-\alpha_t\right) - \mu_t\alpha_t\epsilon_t\bar{M}_t.
$$

The necessary conditions are

$$
\begin{aligned}
\frac{\partial\mathcal{H}_t}{\partial\epsilon_t} &= e^{-rt}\bar{M}_t\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) - \lambda_t\alpha_t\left(1-\alpha_t\right) - \mu_t\alpha_t\bar{M}_t = 0, \\
\dot{\lambda}_t = -\frac{\partial\mathcal{H}_t}{\partial\alpha_t} &= -\left[e^{-rt}\bar{M}_t\left(\epsilon_t sR\right) - \lambda_t\left(\epsilon_t\left(1-\alpha_t\right) - \alpha_t\epsilon_t\right) - \mu_t\epsilon_t\bar{M}_t\right], \\
\dot{\mu}_t = -\frac{\partial\mathcal{H}_t}{\partial\bar{M}_t} &= -\left[e^{-rt}\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right) - \mu_t\alpha_t\epsilon_t\right];
\end{aligned}
$$

laws of motion of the state variables α_t and $\bar{M}_t$,

$$
\dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right) \text{ and}
$$

$$\dot{\bar{M}}_t = -\alpha_t \epsilon_t \bar{M}_t;$$

and the boundary conditions of the state variables.

Define

$$\lambda_t \equiv e^{-rt}\bar{\lambda}_t, \qquad \text{so} \qquad \dot{\lambda}_t = -re^{-rt}\bar{\lambda}_t + e^{-rt}\dot{\bar{\lambda}}_t,$$

$$\mu_t \equiv e^{-rt}\bar{\mu}_t, \qquad \text{so} \qquad \dot{\mu}_t = -re^{-rt}\bar{\mu}_t + e^{-rt}\dot{\bar{\mu}}_t.$$

Using these new costate variables, rewrite the first three necessary conditions:

$$\bar{M}_t\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) - \bar{\lambda}_t\alpha_t\left(1-\alpha_t\right) - \bar{\mu}_t\alpha_t\bar{M}_t = 0,$$

$$-\dot{\bar{\lambda}}_t + \bar{\lambda}_t\epsilon_t\left(1-2\alpha_t\right) = \bar{M}_t\epsilon_t sR - \bar{\mu}_t\epsilon_t\bar{M}_t - r\bar{\lambda}_t,$$

$$-\dot{\bar{\mu}}_t + \bar{\mu}_t\alpha_t\epsilon_t = \alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c - r\bar{\mu}_t.$$

Differentiate the first condition with respect to time:

$$\dot{\bar{M}}_t\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) + \bar{M}_t\left(\dot{\alpha}_t sR - f''\left(\epsilon_t\right)\dot{\epsilon}_t\right)$$
$$+\,\alpha_t\left(1-\alpha_t\right)\left[-\dot{\bar{\lambda}}_t + \bar{\lambda}_t\epsilon_t\left(1-2\alpha_t\right)\right] - \dot{\bar{\mu}}_t\alpha_t\bar{M}_t - \bar{\mu}_t\left[\dot{\alpha}_t\bar{M}_t + \alpha_t\dot{\bar{M}}_t\right] = 0,$$

plug the second condition into it:

$$\dot{\bar{M}}_t\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) + \bar{M}_t\left(\dot{\alpha}_t sR - f''\left(\epsilon_t\right)\dot{\epsilon}_t\right)$$
$$+\,\alpha_t\left(1-\alpha_t\right)\left[\bar{M}_t\epsilon_t sR - \bar{\mu}_t\epsilon_t\bar{M}_t - r\bar{\lambda}_t\right] - \dot{\bar{\mu}}_t\alpha_t\bar{M}_t - \bar{\mu}_t\left[\dot{\alpha}_t\bar{M}_t + \alpha_t\dot{\bar{M}}_t\right] = 0,$$

and simplify

$$\dot{\bar{M}}_t\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) - \bar{M}_t f''\left(\epsilon_t\right)\dot{\epsilon}_t - r\alpha_t\left(1-\alpha_t\right)\bar{\lambda}_t + \alpha_t\bar{M}_t\left[-\dot{\bar{\mu}}_t + \bar{\mu}_t\alpha_t\epsilon_t\right] = 0.$$

Plug the third necessary condition into it:

$$\dot{\bar{M}}_t\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) - \bar{M}_t f''\left(\epsilon_t\right)\dot{\epsilon}_t - r\alpha_t\left(1-\alpha_t\right)\bar{\lambda}_t$$
$$+\,\alpha_t\bar{M}_t\left[\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c - r\bar{\mu}_t\right] = 0.$$

Simplify:

$$\alpha_t\bar{M}_t\left[f'\left(\epsilon_t\right)\epsilon_t - f\left(\epsilon_t\right)\right] - \bar{M}_t f''\left(\epsilon_t\right)\dot{\epsilon}_t - r\left[\alpha_t\left(1-\alpha_t\right)\bar{\lambda}_t + \bar{\mu}_t\alpha_t\bar{M}_t\right] = 0.$$

Finally, use the first necessary condition in the last term to produce:

$$r\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) = \alpha_t\left[f'\left(\epsilon_t\right)\epsilon_t - f\left(\epsilon_t\right)\right] - f''\left(\epsilon_t\right)\dot{\epsilon}_t.$$

Notice that $\bar{M}_t$ was canceled out. Given that $\bar{M}_t > 0$ for any t, this can be done without any problems.

Therefore, the solution to the generalized control problem is represented by the system of first order ordinary differential equations,

$$\begin{cases} \dot{\epsilon}_t = \alpha_t \dfrac{f'\left(\epsilon_t\right)\epsilon_t - f\left(\epsilon_t\right)}{f''\left(\epsilon_t\right)} - r\dfrac{\alpha_t sR - f'\left(\epsilon_t\right) - c}{f''\left(\epsilon_t\right)}, \\ \dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right), \end{cases}$$

and boundary conditions,

$$\alpha_{T_1}, \alpha_{T_2} \text{ given.}$$

Assume, and verify individually for each particular problem, that $\epsilon_t > 0$ for all t between T_1 and T_2. Then, given that α_t monotonically decreases in time, and that time t does not directly influence any equation in the system that characterizes the optimal paths, it is enough to have just one state variable α_t that uniquely characterizes the state at time t. Define the policy function that presents control variable ϵ as a function of state varibale α as $\epsilon\left(\alpha\right)$. Then

$$\epsilon_t \equiv \epsilon\left(\alpha_t\right), \qquad \dot{\epsilon}_t = \epsilon'\left(\alpha_t\right)\dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right)\epsilon'\left(\alpha_t\right).$$

I can now combine two differential equations, drop time subscripts, and produce the first order nonlinear ordinary differential equation that characterizes every solution to the generalized problem in terms of policy function $\epsilon\left(\alpha\right)$:

$$\epsilon'\left(\alpha\right) = -\frac{f'\left(\epsilon\left(\alpha\right)\right)\epsilon\left(\alpha\right) - f\left(\epsilon\left(\alpha\right)\right)}{\left(1-\alpha\right)f''\left(\epsilon\left(\alpha\right)\right)\epsilon\left(\alpha\right)} + r\frac{\alpha sR - f'\left(\epsilon\left(\alpha\right)\right) - c}{\alpha\left(1-\alpha\right)f''\left(\epsilon\left(\alpha\right)\right)\epsilon\left(\alpha\right)}. \tag{I.b}$$

Recall that $f''\left(x\right) > 0$ for all x and that $f\left(\cdot\right)$ is twice continuously differentiable. It means that the right hand side of the expression above is continuously differentiable in ϵ and continuous in α (ignoring its effect on ϵ) for $\alpha \in (0,1)$. If there exist a boundary condition $\epsilon\left(\underline{\alpha}\right) = \underline{\epsilon}$, then according to Picard–Lindelöf theorem the solution to the differential equation above in terms of policy function $\epsilon\left(\alpha\right)$ is unique.

I.B. The Efficient Solution

I.B.1. Existence and Uniqueness

The social planner's problem is

$$\max_{(\epsilon_t, t\geqslant 0)} \int_0^{\infty} e^{-tr} M_t \left[\alpha_t \epsilon_t R - f\left(\epsilon_t\right) - \epsilon_t c\right] \, \mathrm{d}t$$

$$\begin{aligned}
\text{subject to: } & \dot{\alpha}_t = -\alpha_t \epsilon_t \left(1-\alpha_t\right), \\
& \dot{M}_t = -\alpha_t \epsilon_t M_t, \\
& \alpha_0 \text{ given, } \alpha_0 \in [0,1], \\
& M_0 = 1, \\
& \epsilon_t \geqslant 0, \forall t \geqslant 0.
\end{aligned}$$

To write it in the form of the generalized problem, (I.a), solved in Appendix I.A, I set

$$\begin{aligned}
T_1 &= 0, & T_2 &= \infty, \\
\alpha_{T_1} &= \alpha_0, & \bar{M}_t &= M_t, \forall t, \\
s &= 1.
\end{aligned}$$

The only thing left to do is to assume that there exist some finite limit,

$$\lim_{t\to\infty} \alpha_t \equiv \alpha_{\infty},$$

and set $\alpha_{T_2} = \alpha_{\infty}$. Given these assumption, as well as the assumption that experimentation rates will stay positive for all t, I know that all the solutions to the social planner's problem in terms of policy functions must satisfy

$$\epsilon^{*\prime}\left(\alpha\right) = -\frac{f'\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right) - f\left(\epsilon^*\left(\alpha\right)\right)}{\left(1-\alpha\right) f''\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right)} + r\frac{\alpha R - f'\left(\epsilon^*\left(\alpha\right)\right) - c}{\alpha\left(1-\alpha\right) f''\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right)},$$

which is a form of equation (I.b) for given assumptions. I can rewrite the differential equation without fractions:

$$\begin{aligned}
& r\left[\alpha R - f'\left(\epsilon^*\left(\alpha\right)\right) - c\right] \\
&= \alpha\left[f'\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right) - f\left(\epsilon^*\left(\alpha\right)\right) + \left(1-\alpha\right) f''\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right)\epsilon^{*\prime}\left(\alpha\right)\right].
\end{aligned} \tag{I.c}$$

From Appendix I.A, I know that the solution to this differential equation is unique if I manage to establish one boundary condition of the form

$$\epsilon^* \left(\underline{\alpha}\right) = \underline{\epsilon}.$$

To produce the boundary condition, observe that there exists the lowest belief level of α such that it does not make sense to continue with experiments when it is reached. Suppose that $\underline{\alpha} = \frac{c}{R}$. Then

$$\underline{\alpha}\epsilon R - f\left(\epsilon\right) - \epsilon c = \frac{c}{R}\epsilon R - f\left(\epsilon\right) - \epsilon c = -f\left(\epsilon\right) \leqslant 0,$$

which means that the immediate benefits of experimenting are negative for any positive experimentation rate. Posterior belief α can only decrease in time, so if I still decide to experiment ignoring the loss, then in the future I will reach $\alpha < \underline{\alpha}$. My immediate benefits from experimenting will be

$$\alpha\epsilon R - f\left(\epsilon\right) - \epsilon c < \underline{\alpha}\epsilon R - f\left(\epsilon\right) - \epsilon c = -f\left(\epsilon\right) \leqslant 0.$$

Therefore, if the belief level of $\underline{\alpha}$ is reached then there is no reason to carry on with experiments, because it will only cause losses. Notice that it is not the case if $\alpha > \frac{c}{R}$. Suppose that $\alpha = \underline{\alpha} + \delta$, where $\delta > 0$ is very small. Then

$$\alpha\epsilon R - f\left(\epsilon\right) - \epsilon c = \left(\underline{\alpha} + \delta\right)\epsilon R - f\left(\epsilon\right) - \epsilon c = \delta\epsilon R - f\left(\epsilon\right),$$

and it is always possible to find ϵ to keep this expression positive. Therefore, the boundary condition exists and it is

$$\epsilon^* \left(\underline{\alpha}\right) = \epsilon^* \left(\frac{c}{R}\right) = 0.$$

There is only one solution to the social planner's problem in terms of policy functions. I can also establish that for the social planner's problem, $\alpha_\infty = \frac{c}{R}$. This condition is verified in the next Appendix.

I.B.2. Properties of the Efficient Experimentation Path

The important properties of the efficient experimentation path are:

Staticity at the top : the experimentation rate for sure projects ($\alpha = 1$) is stationary. For belief level $\alpha = 1$, differential equation (I.c) degenerates to equation

$$r\left[R - f'\left(\epsilon^*\left(1\right)\right) - c\right] = f'\left(\epsilon^*\left(1\right)\right)\epsilon^*\left(1\right) - f\left(\epsilon^*\left(1\right)\right).$$

If $\alpha_0 = 1$, then for all t, $\alpha_t = 1$ as well:

$$\alpha_t = \frac{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}} = \frac{e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}} = 1.$$

In this case, efficient experimentation level $\epsilon^*\left(1\right)$ is stationary, monotone, and strictly positive provided that $R > c$ (so that $\frac{c}{R} < 1$). Hence, sure projects must be worked on at the constant efficient experimentation rate, $\epsilon^*\left(1\right)$, until success happens.

Experimentation rate strictly decreases in time : for $\alpha \in \left(\frac{c}{R}, 1\right)$, the experimentation rate *increases* in α, but since thr posterior belief, α, strictly decreases over time given no success[1], the experimentation rate decreases in time. If $\epsilon^{*\prime}\left(\alpha\right) > 0$, it means that

$$\dot{\epsilon}^*\left(\alpha\right) = \dot{\alpha}\epsilon^{*\prime}\left(\alpha\right) = -\alpha\epsilon^*\left(\alpha\right)\left(1-\alpha\right)\epsilon^{*\prime}\left(\alpha\right) < 0.$$

Imagine that, on the contrary, that is not the case. Solutions to the first order differential equations are continuously differentiable. Since $\epsilon^*\left(\frac{c}{R}\right) = 0$ and $\epsilon^*\left(1\right) > 0$, given $\frac{c}{R} < 1$, experimentation rate $\epsilon^*\left(\alpha\right)$ must continuously increase somewhere for α between $\frac{c}{R}$ and 1. So, for it to ever *decrease*, function $\epsilon^*\left(\alpha\right)$ must have at least one extreme point. Suppose there is such a point. Pick some $\hat{\alpha}$, differentiate both sides of (I.c) with respect to α and assume that $\epsilon^{*\prime}\left(\hat{\alpha}\right) = 0$:

$$rR = f'\left(\epsilon^*\left(\hat{\alpha}\right)\right)\epsilon^*\left(\hat{\alpha}\right) - f\left(\epsilon^*\left(\hat{\alpha}\right)\right) + \hat{\alpha}\left(1-\hat{\alpha}\right)f''\left(\epsilon^*\left(\hat{\alpha}\right)\right)\epsilon^*\left(\hat{\alpha}\right)\epsilon^{*\prime\prime}\left(\hat{\alpha}\right).$$

Next, plug $\epsilon^{*\prime}\left(\hat{\alpha}\right) = 0$ into (I.c) directly:

$$r\left[\hat{\alpha}R - f'\left(\epsilon^*\left(\hat{\alpha}\right)\right) - c\right] = \hat{\alpha}\left[f'\left(\epsilon^*\left(\hat{\alpha}\right)\right)\epsilon^*\left(\hat{\alpha}\right) - f\left(\epsilon^*\left(\hat{\alpha}\right)\right)\right].$$

[1] Recall that $\frac{\mathrm{d}\alpha}{\mathrm{d}t} \equiv \dot{\alpha} = -\alpha\epsilon\left(1-\alpha\right) < 0$.

Combine these two conditions together:

$$rR = \frac{r\left[\hat{\alpha}R - f'\left(\epsilon^*\left(\hat{\alpha}\right)\right) - c\right]}{\hat{\alpha}} + \hat{\alpha}\left(1-\hat{\alpha}\right) f''\left(\epsilon^*\left(\hat{\alpha}\right)\right) \epsilon^*\left(\hat{\alpha}\right) \epsilon^{*\prime\prime}\left(\hat{\alpha}\right),$$

and simplify:

$$r\frac{f'\left(\epsilon^*\left(\hat{\alpha}\right)\right) + c}{\hat{\alpha}} = \hat{\alpha}\left(1-\hat{\alpha}\right) f''\left(\epsilon^*\left(\hat{\alpha}\right)\right) \epsilon^*\left(\hat{\alpha}\right) \epsilon^{*\prime\prime}\left(\hat{\alpha}\right).$$

Given that $\hat{\alpha} \in \left(\frac{c}{R}, 1\right)$, $\epsilon^* \geqslant 0$, $f'(\cdot) \geqslant 0$, and $f''(\cdot) > 0$, it must be the case that

$$\epsilon^{*\prime\prime}\left(\hat{\alpha}\right) > 0$$

when $\epsilon^{*\prime}\left(\hat{\alpha}\right) = 0$. It means that *all* the extreme points that can be found along the efficient experimentation path must be local minima. It is only possible if $\epsilon^*(\alpha)$ first strictly decreases and then strictly increases on $\left(\frac{c}{R}, 1\right)$, but this is impossible: $\epsilon^*\left(\frac{c}{R}\right) = 0$, and experimentation rates cannot be negative. I reached a contradiction. Therefore, there are no extreme points and function $\epsilon^*(\alpha)$ strictly increases in α, and so it strictly decreases over time.

Experimenting never stops : despite that experimentation rate strictly decreases over time, experiments never stop. To show that experiments never stop, I demonstrate that for $\alpha \in \left[\frac{c}{R}, 1\right]$ efficient policy function $\epsilon^*(\alpha)$ is everywhere below critical function $\bar{\epsilon}(\alpha)$ from Appendix C and that $\epsilon^*(\alpha) > 0$ for all $\alpha \in \left(\frac{c}{R}, 1\right]$:

- function $\epsilon^*(\alpha)$ is a solution to an ordinary differential equation, it is continuous;
- $\epsilon^*(\alpha) > 0$ on $\left(\frac{c}{R}, 1\right]$ as it is strictly increasing from $\epsilon\left(\frac{c}{R}\right) = 0$ to $\epsilon^*(1) < \infty$, so it is bounded.

The only condition that must be verified is $\epsilon^{*\prime}\left(\underline{\alpha}\right) < \infty$. Express $\epsilon^{*\prime}\left(\underline{\alpha}\right)$ from (I.c) and take limits:

$$\begin{aligned}
\epsilon^{*\prime}\left(\frac{c}{R}\right) &= \lim_{\alpha \to \frac{c}{R}} \frac{r\left[\alpha R - f'\left(\epsilon^*(\alpha)\right) - c\right] - \alpha f'\left(\epsilon^*(\alpha)\right)\epsilon^*(\alpha) + \alpha f\left(\epsilon^*(\alpha)\right)}{\alpha\left(1-\alpha\right) f''\left(\epsilon^*(\alpha)\right)\epsilon^*(\alpha)} \\
&= \frac{rR^2\left[\frac{R}{\epsilon^{*\prime}\left(\frac{c}{R}\right)} - f''(0)\right]}{c\left(R-c\right) f''(0)}.
\end{aligned}$$

Assuming $\epsilon^{*\prime}\left(\frac{c}{R}\right) = \infty$ will be inconsistent with this expression, so it must be the case that $\epsilon^{*\prime}\left(\frac{c}{R}\right) < \infty$.

Therefore, the efficient experimentation rate strictly decreases indefinitely over time until the project is a success or forever. This confirms the assumption that for all $t \geqslant 0$, $\epsilon_t^* > 0$, thus α_t uniquely characterizes the state.

I.C. The Equilibrium with the Binding Budget Constraint

I.C.1. Phase Two of the Game

The original problem is

$$
\begin{aligned}
&\max_{(\epsilon_t, t \geqslant 0)} \left[P + \int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t s R - f(\epsilon_t) - \epsilon_t c \right] \mathrm{d}t \right] \\
\text{subject to: } & P = \int_0^\infty e^{-tr} \epsilon_t c \, \mathrm{d}t, \\
& \dot{\alpha}_t = -\alpha_t \epsilon_t (1 - \alpha_t), \\
& \dot{M}_t = -\alpha_t \epsilon_t M_t, \\
& P, s \text{ given}, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\
& M_0 = 1, \\
& \epsilon_t \geqslant 0, \forall t \geqslant 0.
\end{aligned}
$$

Use λ as a multiplier for the first constraint, drop constant P from the objective function, and rewrite the problem:

$$
\begin{aligned}
&\max_{(\epsilon_t, t \geqslant 0), \lambda} \left[\int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t s R - f(\epsilon_t) - \epsilon_t c \right] \mathrm{d}t + \lambda \left(P - \int_0^\infty e^{-tr} \epsilon_t c \, \mathrm{d}t \right) \right] \\
\text{subject to: } & \dot{\alpha}_t = -\alpha_t \epsilon_t (1 - \alpha_t), \\
& \dot{M}_t = -\alpha_t \epsilon_t M_t, \\
& P, s \text{ given}, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\
& M_0 = 1, \\
& \epsilon_t \geqslant 0, \forall t \geqslant 0.
\end{aligned}
$$

Since

$$\int_0^\infty e^{-tr}\,\mathrm{d}t = \frac{1}{r},$$

it is possible to substitute

$$P\frac{r}{r} = \int_0^\infty e^{-tr} rP\,\mathrm{d}t$$

for P and rewrite the problem

$$\begin{aligned}
\max_{(\epsilon_t, t\geqslant 0),\lambda} & \int_0^\infty e^{-tr}\left(\lambda rP - \lambda\epsilon_t c + M_t\left[\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right]\right)\,\mathrm{d}t \\
\text{subject to: } & \dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right), \\
& \dot{M}_t = -\alpha_t\epsilon_t M_t, \\
& P, s \text{ given}, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0,1], \\
& M_0 = 1, \\
& \epsilon_t \geqslant 0, \forall t \geqslant 0.
\end{aligned}$$

I use the Pontryagin's Maximum Principle to solve this problem. Assign Lagrange multipliers μ_t and ν_t to the first and the second constraints, respectively, and rewrite the problem in the Lagrangian form:

$$\begin{aligned}
\mathcal{L} = & \int_0^\infty e^{-tr}\left(\lambda rP - \lambda\epsilon_t c + M_t\left[\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right]\right)\,\mathrm{d}t \\
& + \int_0^\infty \left[\mu_t\left(-\alpha_t\epsilon_t\left(1-\alpha_t\right) - \dot{\alpha}_t\right) + \nu_t\left(-\alpha_t\epsilon_t M_t - \dot{M}_t\right)\right]\,\mathrm{d}t \\
& \alpha_0 \text{ given}, \\
& M_0 = 1.
\end{aligned}$$

Assume that limits $\lim_{t\to\infty}\alpha_t \equiv \alpha_\infty < 0$ and $\lim_{t\to\infty} M_t \equiv M_\infty < 0$ exist. Integrate by parts $\mu_t\dot{\alpha}_t$ and $\nu_t\dot{M}_t$ and rewrite the problem:

$$\mathcal{L} = \int_0^\infty \left[e^{-tr}\left(\lambda rP - \lambda\epsilon_t c + M_t\left[\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right]\right) - \mu_t\alpha_t\epsilon_t\left(1-\alpha_t\right) - \nu_t\alpha_t\epsilon_t M_t\right]\,\mathrm{d}t$$

$$+\int_0^\infty \left[\dot{\mu}_t\alpha_t + \dot{\nu}_t M_t\right] \,\mathrm{d}t - \left[\mu_\infty\alpha_\infty - \mu_0\alpha_0\right] - \left[\nu_\infty M_\infty - \mu_0\right].$$

Define Hamiltonian function

$$\mathcal{H}_t = e^{-tr}\left(\lambda r P - \lambda\epsilon_t c + M_t\left[\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right]\right) - \mu_t\alpha_t\epsilon_t\left(1-\alpha_t\right) - \nu_t\alpha_t\epsilon_t M_t.$$

The necessary conditions are

$$\begin{aligned}
\frac{\partial\mathcal{H}_t}{\partial\epsilon_t} &= e^{-tr}\left(-\lambda c + M_t\left[\alpha_t sR - f'\left(\epsilon_t\right) - c\right]\right) - \mu_t\alpha_t\left(1-\alpha_t\right) - \nu_t\alpha_t M_t = 0, \\
\dot{\mu}_t = -\frac{\partial\mathcal{H}_t}{\partial\alpha_t} &= -\left[e^{-tr}M_t\epsilon_t sR - \mu_t\epsilon_t\left(1-2\alpha_t\right) - \nu_t\epsilon_t M_t\right], \\
\dot{\nu}_t = -\frac{\partial\mathcal{H}_t}{\partial M_t} &= -\left[e^{-tr}\left(\alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c\right) - \nu_t\alpha_t\epsilon_t\right];
\end{aligned}$$

laws of motion of the state variables α_t and M_t,

$$\dot{\alpha}_t = -\alpha_t\epsilon_t\left(1-\alpha_t\right) \text{ and}$$

$$\dot{M}_t = -\alpha_t\epsilon_t M_t;$$

and the boundary conditions of the state variables.

Define

$$\begin{aligned}
\mu_t &\equiv e^{-rt}\bar{\mu}_t, &&\text{so} && \dot{\mu}_t = -re^{-rt}\bar{\mu}_t + e^{-rt}\dot{\bar{\mu}}_t, \\
\nu_t &\equiv e^{-rt}\bar{\nu}_t, &&\text{so} && \dot{\nu}_t = -re^{-rt}\bar{\nu}_t + e^{-rt}\dot{\bar{\nu}}_t.
\end{aligned}$$

Using these new costate variables, rewrite the first three necessary conditions:

$$-\lambda c + M_t\left[\alpha_t sR - f'\left(\epsilon_t\right) - c\right] - \bar{\mu}_t\alpha_t\left(1-\alpha_t\right) - \bar{\nu}_t\alpha_t M_t = 0,$$

$$-\dot{\bar{\mu}}_t + \bar{\mu}_t\epsilon_t\left(1-2\alpha_t\right) = M_t\epsilon_t sR - \bar{\nu}_t\epsilon_t M_t - r\bar{\mu}_t,$$

$$-\dot{\bar{\nu}}_t + \bar{\nu}_t\alpha_t\epsilon_t = \alpha_t\epsilon_t sR - f\left(\epsilon_t\right) - \epsilon_t c - r\bar{\nu}_t.$$

Differentiate the first condition with respect to time:

$$\dot{M}_t\left(\alpha_t sR - f'\left(\epsilon_t\right) - c\right) + M_t\left(\dot{\alpha}_t sR - f''\left(\epsilon_t\right)\dot{\epsilon}_t\right)$$
$$+ \alpha_t\left(1-\alpha_t\right)\left[-\dot{\bar{\mu}}_t + \bar{\mu}_t\epsilon_t\left(1-2\alpha_t\right)\right] - \dot{\bar{\nu}}_t\alpha_t M_t - \bar{\nu}_t\left[\dot{\alpha}_t M_t + \alpha_t\dot{M}_t\right] = 0.$$

Using Appendix I.A as a guideline, I produce:

$$\alpha_t M_t \left[f'(\epsilon_t)\epsilon_t - f(\epsilon_t)\right] - M_t f''(\epsilon_t)\dot{\epsilon}_t - r\left[\alpha_t(1-\alpha_t)\bar{\mu}_t + \bar{\nu}_t \alpha_t M_t\right] = 0.$$

Finally, insert the first necessary condition into the last term:

$$r\left[-\lambda c + M_t\left(\alpha_t sR - f'(\epsilon_t) - c\right)\right] = \alpha_t M_t\left[f'(\epsilon_t)\epsilon_t - f(\epsilon_t)\right] - M_t f''(\epsilon_t)\dot{\epsilon}_t,$$

or simply

$$r\left[\alpha_t sR - f'(\epsilon_t) - c\left(1 + \frac{\lambda}{M_t}\right)\right] = \alpha_t\left[f'(\epsilon_t)\epsilon_t - f(\epsilon_t)\right] - f''(\epsilon_t)\dot{\epsilon}_t.$$

written in the policy function form, it becomes:

$$r\left[\alpha sR - f'(\epsilon(\alpha)) - c\left(1 + \frac{\lambda}{M}\right)\right]$$
$$= \alpha\left[f'(\epsilon(\alpha))\epsilon(\alpha) - f(\epsilon(\alpha)) + (1-\alpha)f''(\epsilon(\alpha))\epsilon(\alpha)\epsilon'(\alpha)\right].$$

Given that

$$M = \frac{1-\alpha_0}{1-\alpha},$$

this differential equation can be written as

$$r\left[\alpha sR - f'(\epsilon(\alpha)) - c\left(1 + \lambda\frac{1-\alpha}{1-\alpha_0}\right)\right]$$
$$= \alpha\left[f'(\epsilon(\alpha))\epsilon(\alpha) - f(\epsilon(\alpha)) + (1-\alpha)f''(\epsilon(\alpha))\epsilon(\alpha)\epsilon'(\alpha)\right].$$

Comparing it to the differential equation that describes the equilibrium experimentation rate when the budget constraint does not bind,

$$r\left[\alpha sR - f'(\epsilon(\alpha)) - c\right] = \alpha\left[f'(\epsilon(\alpha))\epsilon(\alpha) - f(\epsilon(\alpha)) + (1-\alpha)f''(\epsilon(\alpha))\epsilon(\alpha)\epsilon'(\alpha)\right],$$

it is easy to see that for all $\alpha \in \left[\frac{c}{sR}, \alpha_0\right]$, the policy function corresponding to the binding budget constraint will be lower than the policy function when the budget constraint does not bind. This is because $\lambda > 0$, $\alpha \leqslant \alpha_0$, and thus

$$c\left(1 + \lambda\frac{1-\alpha}{1-\alpha_0}\right) > c.$$

The effect of having a binding constraint is similar to having an increase in the marginal cost of experimentation c for every level of $\alpha \leqslant \alpha_0$. According to Appendix I.D.2, it means that the policy function associated with the binding budget constraint will be below the policy function with no constraint.

I.D. Comparative Statics

I.D.1. Policy Function Effects

I show that, given two different policy functions—one is everywhere higher than the other—the total expected revenue from experimenting is higher for the former than for the later.

I begin by showing that given two continuously differentiable policy functions, $\epsilon\left(\alpha\right)$ and $\hat{\epsilon}\left(\alpha\right)$, such that for all α,

$$\epsilon\left(\alpha\right) > \hat{\epsilon}\left(\alpha\right),$$

the total expected revenue from experimentation generated by using policy function $\epsilon\left(\alpha\right)$ is higher than the expected revenue generated by adhering to policy function $\hat{\epsilon}\left(\alpha\right)$. If it is so, then it must be the case that

$$\int_0^\infty e^{-tr-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\alpha_t\epsilon_t R\,\mathrm{d}t > \int_0^\infty e^{-tr-\int_0^t \hat{\alpha}_\tau\hat{\epsilon}_\tau\,\mathrm{d}\tau}\hat{\alpha}_t\hat{\epsilon}_t R\,\mathrm{d}t,$$

where α_t and ϵ_t are the time-paths of the posterior beliefs and the experimentation rates produced by the first policy function, $\epsilon\left(\alpha\right)$, and $\hat{\alpha}_t$, $\hat{\epsilon}_t$ are the respective time-paths produced by the second policy function, $\hat{\epsilon}\left(\alpha\right)$. Using integration by parts, I derive

$$\begin{aligned}\int_0^\infty e^{-tr-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\alpha_t\epsilon_t\,\mathrm{d}t &= -\int_0^\infty e^{-tr}\frac{\mathrm{d}}{\mathrm{d}t}e^{-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\,\mathrm{d}t \\ &= -\left. e^{-tr}e^{-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\right|_0^\infty - r\int_0^\infty e^{-tr}e^{-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\,\mathrm{d}t \\ &= 1 - r\int_0^\infty e^{-tr}e^{-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau}\,\mathrm{d}t.\end{aligned}$$

From Appendix A, I know that

$$e^{-\int_0^t \alpha_\tau\epsilon_\tau\,\mathrm{d}\tau} = 1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau\,\mathrm{d}\tau},$$

therefore

$$\begin{aligned}\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \alpha_t \epsilon_t \, \mathrm{d}t &= 1 - r \int_0^\infty e^{-tr} e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \, \mathrm{d}t \\ &= 1 - r \int_0^\infty e^{-tr} \left(1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}\right) \mathrm{d}t \\ &= 1 - (1 - \alpha_0) \, r \int_0^\infty e^{-tr} \, \mathrm{d}t - r\alpha_0 \int_0^\infty e^{-tr} e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} \, \mathrm{d}t \\ &= \alpha_0 \left(1 - r \int_0^\infty e^{-tr} e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} \, \mathrm{d}t\right).\end{aligned}$$

Given that

$$\alpha_t \equiv \frac{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}},$$

$$e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} = \frac{1 - \alpha_0}{\alpha_0} \frac{\alpha_t}{1 - \alpha_t},$$

and hence

$$\begin{aligned}\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \alpha_t \epsilon_t \, \mathrm{d}t &= \alpha_0 \left(1 - r \int_0^\infty e^{-tr} e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau} \, \mathrm{d}t\right) \\ &= \alpha_0 \left(1 - r \frac{1 - \alpha_0}{\alpha_0} \int_0^\infty e^{-tr} \frac{\alpha_t}{1 - \alpha_t} \, \mathrm{d}t\right) \\ &= \alpha_0 - r \left(1 - \alpha_0\right) \int_0^\infty e^{-tr} \frac{\alpha_t}{1 - \alpha_t} \, \mathrm{d}t.\end{aligned}$$

Therefore, to establish that

$$\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \alpha_t \epsilon_t R \, \mathrm{d}t > \int_0^\infty e^{-tr-\int_0^t \hat{\alpha}_\tau \hat{\epsilon}_\tau \, \mathrm{d}\tau} \hat{\alpha}_t \hat{\epsilon}_t R \, \mathrm{d}t,$$

it is enough to confirm that

$$\alpha_0 R - r \left(1 - \alpha_0\right) R \int_0^\infty e^{-tr} \frac{\alpha_t}{1 - \alpha_t} \, \mathrm{d}t > \alpha_0 R - r \left(1 - \alpha_0\right) R \int_0^\infty e^{-tr} \frac{\hat{\alpha}_t}{1 - \hat{\alpha}_t} \, \mathrm{d}t,$$

or that

$$\int_0^\infty e^{-tr} \frac{\hat{\alpha}_t}{1 - \hat{\alpha}_t} \, \mathrm{d}t > \int_0^\infty e^{-tr} \frac{\alpha_t}{1 - \alpha_t} \, \mathrm{d}t.$$

Comparing pointwise, what is needed to be shown is that for all $t > 0$,

$$\frac{\hat{\alpha}_t}{1 - \hat{\alpha}_t} > \frac{\alpha_t}{1 - \alpha_t},$$

or

$$\hat{\alpha}_t > \alpha_t.$$

It means that the belief evolution path associated with the "lower" policy function always stays above the belief evolution path associated with the "higher" policy function. Simply put, the experimenting agent stays relatively more optimistic longer when faced with the lower policy function.

Given that the prior is α_0, at $t = 0$,

$$\alpha_t = \hat{\alpha}_t = \alpha_0,$$

however, the time-derivatives of the posterior believes for two policy functions are different:

$$-\hat{\alpha}_t \hat{\epsilon}\left(\hat{\alpha}_t\right)\left(1-\hat{\alpha}_t\right) = -\alpha_0 \hat{\epsilon}\left(\alpha_0\right)\left(1-\alpha_0\right) > -\alpha_0 \epsilon\left(\alpha_0\right)\left(1-\alpha_0\right) = \alpha_t \epsilon\left(\alpha_t\right)\left(1-\alpha_t\right).$$

Due to continuity of the belief evolution paths, it means that for some small $\bar{t} > 0$, for all $t \in (0, \bar{t})$,

$$\hat{\alpha}_t > \alpha_t$$

because α_t decreases faster than $\hat{\alpha}_t$.

Suppose that for some $t' > \bar{t}$, this inequality reverses. Due to continuity it means that there must be some point at which belief levels coincide. Suppose that this happens exactly at $t = \bar{t}$:

$$\hat{\alpha}_{\bar{t}} = \alpha_{\bar{t}}.$$

However, then the time-derivatives of the posterior beliefs at $t = \bar{t}$ will be different again:

$$-\hat{\alpha}_{\bar{t}} \hat{\epsilon}\left(\hat{\alpha}_{\bar{t}}\right)\left(1-\hat{\alpha}_{\bar{t}}\right) > -\alpha_{\bar{t}} \epsilon\left(\alpha_{\bar{t}}\right)\left(1-\alpha_{\bar{t}}\right)$$

simply because for $\hat{\alpha}_{\bar{t}} = \alpha_{\bar{t}}$,

$$\hat{\epsilon}\left(\hat{\alpha}_{\bar{t}}\right) < \epsilon\left(\alpha_{\bar{t}}\right).$$

Due to continuity, it means that there must be some $\underline{t} < \bar{t}$, such that for all $t \in (\underline{t}, \bar{t})$,

$$\alpha_t > \hat{\alpha}_t.$$

This will contradict the fact that for all $t \in (0, \bar{t})$,

$$\hat{\alpha}_t > \alpha_t,$$

therefore, $\bar{t}$ with the desired properties does not exist and $\hat{\alpha}_t$ is everywhere strictly above α_t for $t > 0$. I established that a higher policy function implies higher total expected revenue from experimenting.

I.D.2. Parameter Changes Effects

There are four important parameters that affect a policy function: total surplus R, marginal monetary cost of experimenting c, discount coefficient r, and prior probability that the project is good α_0. There is another parameter that affects the movement of a policy function, which is itself affected by the changes in these main parameters, equilibrium share s^{**}. Thus any parameter change must be analyzed together with the resulting change in share s^{**}.

First, consider an increase in surplus R. Consider the budget constraint:

$$\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau^{**}\, \mathrm{d}\theta} \alpha_t \epsilon_t^{**} (1-s) R \, \mathrm{d}t \geqslant \int_0^\infty e^{-tr} \epsilon_t^{**} c \, \mathrm{d}t.$$

This constraint was binding prior to the increase in the surplus size. Remember that the entrepreneur is ex ante indifferent to the size of the share, it only affects the budget constraint and the ex post incentives to experiment. So suppose that the entrepreneur adjusts s such that sR does not change after the increase in R. It means that the ex post incentives of the entrepreneur are the same: she will experiment according to the same policy function as before the change in R, so path ϵ_t^{**} is unaffected. However, sR staying the same implies that $(1-s) R$ increases as R raises. Then the left had side of the budget constraint expands while the right hand side stays constant due to unaffected policy function. The constraint no longer binds. As it was argued in the description of the first phase of the game with non-binding budget constraint, when the constrain does not bind, Entrepreneur wants to raise share s to improve the ex post incentives to experiment. Therefore, as R moves up, sR increases as well.

Consider two policy functions $\epsilon^{**}(\alpha)$ and $\varepsilon^{**}(\alpha)$, the first one is associated with surplus R and the second—with surplus $\hat{R}$, $\hat{R} > R$. Share $\hat{s}$ corresponds to surplus $\hat{R}$ and share s—to surplus R. Therefore, $\hat{s}\hat{R} > sR$. Now, consider a point where a policy function is zero. Given that $\hat{s}\hat{R} > sR$, $\frac{c}{sR} < \frac{c}{\hat{s}\hat{R}}$, which means that policy function $\varepsilon^{**}(\alpha)$ intersects horizontal axis to the left of $\epsilon^{**}(\alpha)$. Policy functions are continuous and strictly increasing, so it must be the case that for some $\hat{\alpha}$, for all $\alpha < \hat{\alpha}$, $\varepsilon^{**}(\alpha) > \epsilon^{**}(\alpha)$. Suppose that for $\hat{\alpha}$, $\varepsilon^{**}(\hat{\alpha}) = \epsilon^{**}(\hat{\alpha}) = x$. Then it must be the case that (see (I.3))

$$\begin{aligned}\varepsilon^{**\prime}(\hat{\alpha}) &= r\frac{\hat{\alpha}\hat{s}\hat{R} - f'(x) - c - \hat{\alpha}f'(x)x + \hat{\alpha}f(x)}{\hat{\alpha}(1-\hat{\alpha})f''(x)x} \\ &> r\frac{\hat{\alpha}sR - f'(x) - c - \hat{\alpha}f'(x)x + \hat{\alpha}f(x)}{\hat{\alpha}(1-\hat{\alpha})f''(x)x} = \epsilon^{**\prime}(\hat{\alpha}).\end{aligned}$$

By intermediate value theorem, there must exist some small $\delta > 0$, such that

$$\varepsilon^{**\prime}(\hat{\alpha}-\delta) \geqslant \epsilon^{**\prime}(\hat{\alpha}-\delta).$$

However, given the assumption that for $\alpha < \hat{\alpha}$, $\varepsilon^{**}(\alpha) > \epsilon^{**}(\alpha)$, it would imply that

$$\varepsilon^{**}(\hat{\alpha}) > \epsilon^{**}(\hat{\alpha}).$$

There two policy functions never intersect. It means that $\varepsilon^{**}(\alpha)$ must be everywhere above $\epsilon^{**}(\alpha)$. Increasing surplus R shifts policy curve upward.

Second, using similar logic, it is easy to conclude that increasing the marginal monetary cost of experimentation, c, shifts the policy curve downward. Consider a hike in c. Suppose the policy function shifts upward or stays same as before the change. An increase in c coupled with nondecreasing rates of experimentations will result in the budget constraint no longer being satisfied. To improve the situation, the entrepreneur will have to decrease share s. Thus share will decrease.

Now, consider two policy functions: $\epsilon^{**}(\alpha)$, which is associated with costs c and share s; and $\varepsilon^{**}(\alpha)$, which corresponds to surplus $\hat{c} > c$ and share $\hat{s} < s$. Consider the point where a policy function is zero. Given $\hat{c} > c$ and $\hat{s} < s$, $\frac{c}{sR} < \frac{\hat{c}}{\hat{s}R}$, so the point where policy function $\varepsilon^{**}(\alpha)$ is zero is to the right of the point where $\epsilon^{**}(\alpha) = 0$. It can not be to the left of it, it will imply that the policy function for higher cost is everywhere above the policy function for the lower cost, which is impossible. Efficient policy functions are continuous and strictly increasing, so it must be the case that for some $\hat{\alpha}$, for all $\alpha < \hat{\alpha}$, $\varepsilon^{**}(\alpha) < \epsilon^{**}(\alpha)$. Suppose that for $\hat{\alpha}$,

$\varepsilon^{**}(\hat{\alpha}) = \epsilon^{**}(\hat{\alpha}) = x$. Then it must be the case that

$$\begin{aligned}\varepsilon^{**\prime}(\hat{\alpha}) &= r\frac{\hat{\alpha}\hat{s}R - f'(x) - \hat{c} - \hat{\alpha}f'(x)x + \hat{\alpha}f(x)}{\hat{\alpha}(1-\hat{\alpha})f''(x)x} \\ &< r\frac{\hat{\alpha}sR - f'(x) - c - \hat{\alpha}f'(x)x + \hat{\alpha}f(x)}{\hat{\alpha}(1-\hat{\alpha})f''(x)x} = \epsilon^{**\prime}(\hat{\alpha}).\end{aligned}$$

By intermediate value theorem, there must exist some small $\delta > 0$, such that

$$\varepsilon^{**\prime}(\hat{\alpha} - \delta) \leqslant \epsilon^{**\prime}(\hat{\alpha} - \delta).$$

However, given the assumption that for $\alpha < \hat{\alpha}$, $\varepsilon^{**}(\alpha) < \epsilon^{**}(\alpha)$,

$$\varepsilon^{**}(\hat{\alpha}) < \epsilon^{**}(\hat{\alpha}).$$

It means that policy function $\varepsilon^{**}(\alpha)$ is everywhere below $\epsilon^{**}(\alpha)$. Thus increasing marginal monetary costs c shifts policy curve downward.

Third, an increase in the discount rate, other things equal, means that the distant future costs, which are high compared to the distant future benefits, will be given a lower weight. At the same time, future benefits, which are closer to time zero and are high compared to the costs, will be given higher weight. In other words, the budget constraint will become more relaxed. Additionally, since increasing r improves experimentation rates for high belief levels disproportionately more than experimentation rates for lower belief levels, the experimentation rates closer to time zero will increase and so the budget constraint might become even more relaxed. This will result in the possibility of increasing the share, which the entrepreneur will use. So as the result of the discount rate hikes the equilibrium share will increase.

Again, pick two policy functions, $\epsilon^{**}(\alpha)$ and $\varepsilon^{**\prime}(\alpha)$, associated, respectively, with discount rates r and $\hat{r} > r$. The discount rate alone does not affect the point where policy functions are equal to zero, but since share $\hat{s}$ associated with function $\varepsilon^{**\prime}(\alpha)$ is higher than share s associated with policy function $\epsilon^{**}(\alpha)$, $\frac{c}{sR} < \frac{c}{\hat{s}R}$. Thus policy function $\varepsilon^{**}(\alpha)$ intersects horizontal axis to the left of $\epsilon^{**}(\alpha)$. Policy functions are continuous and strictly increasing, so it must be true that for some $\hat{\alpha}$, for all $\alpha < \hat{\alpha}$, $\varepsilon^{**}(\alpha) > \epsilon^{**}(\alpha)$. Again, suppose that for $\hat{\alpha}$, $\varepsilon^{**}(\hat{\alpha}) = \epsilon^{**}(\hat{\alpha}) = x$. Then it must be true that

$$\varepsilon^{**\prime}(\hat{\alpha}) = \hat{r}\frac{\hat{\alpha}\hat{s}R - f'(x) - c - \hat{\alpha}f'(x)x + \hat{\alpha}f(x)}{\hat{\alpha}(1-\hat{\alpha})f''(x)x}$$

$$> r\frac{\hat{\alpha} sR - f'(x) - c - \hat{\alpha} f'(x) x + \hat{\alpha} f(x)}{\hat{\alpha}(1-\hat{\alpha}) f''(x) x} = \epsilon^{**\prime}(\hat{\alpha}).$$

By intermediate value theorem, there exist some tiny $\delta > 0$, such that

$$\varepsilon^{**\prime}(\hat{\alpha} - \delta) \geqslant \epsilon^{**\prime}(\hat{\alpha} - \delta).$$

Given the assumption that for $\alpha < \hat{\alpha}$, $\varepsilon^{**}(\alpha) > \epsilon^{**}(\alpha)$, it means that

$$\varepsilon^{**}(\hat{\alpha}) > \epsilon^{**}(\hat{\alpha}).$$

Thus policy function $\varepsilon^{**}(\alpha)$ is everywhere above $\epsilon^{**}(\alpha)$. Increasing discount coefficient r shifts policy functions upward.

Finally, starting with a higher prior means that the entrepreneur will need more capital to carry on with experiments at a higher rate longer. At the same time, it implies the possibility to conduct experiments at high experimentation rates and receive higher expected surplus. So the share may go in any direction depending on the parameters of the model, it may increase or decrease as the prior goes up, but eventually, it will decline to exactly one-half as the prior reaches $\frac{2c}{R}$. Therefore, the effect of the change in the prior belief that the project is good on the policy functions is ambiguous.

I.E. Audited Reporting in Crowdfunding

It is easy to show that committing to reach aggregate efficient levels of experimentation every reporting period is inefficient. Since the optimal experimentation path from the system of differential equations is

$$\dot{\epsilon}_t = \alpha_t \frac{f'(\epsilon_t)\epsilon_t - f(\epsilon_t)}{f''(\epsilon_t)} - r\frac{\alpha_t sR - f'(\epsilon_t) - c}{f''(\epsilon_t)},$$

while the efficient path is described by

$$\dot{\epsilon^*}_t = \alpha_t \frac{f'(\epsilon_t^*)\epsilon_t^* - f(\epsilon_t^*)}{f''(\epsilon_t^*)} - r\frac{\alpha_t R - f'(\epsilon_t^*) - c}{f''(\epsilon_t^*)},$$

then the two paths can only be the same if $s = 1$. This is impossible because when the share equals one, the investors will not benefit from the project at all, and thus will not be willing to donate anything. The entrepreneur needs to offer something to the backers in return for their contributions. Therefore, periodic audited disclosure does not restore full efficiency.

Periodic audited disclosure, however, can be Pareto improving when the entrepreneur commits to reach the same posterior belief levels as in the efficient case over equally displaced intervals of time. Consider the entrepreneur's maximization problem at phase one of the basic game, the game without audit or reporting:

$$\begin{aligned}
&\max_{s} \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \left[\alpha_t \epsilon_t R - f\left(\epsilon_t\right) - \epsilon_t c\right] \,\mathrm{d}t \\
\text{subject to: } &\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \alpha_t \epsilon_t \left(1-s\right) R \,\mathrm{d}t = \int_0^\infty e^{-tr} \epsilon_t c \,\mathrm{d}t, \\
&\dot{\epsilon}_t = \alpha_t \frac{f'\left(\epsilon_t\right)\epsilon_t - f\left(\epsilon_t\right)}{f''\left(\epsilon_t\right)} - r\frac{\alpha_t s R - f'\left(\epsilon_t\right) - c}{f''\left(\epsilon_t\right)}, \\
&\dot{\alpha}_t = -\alpha_t \epsilon_t \left(1-\alpha_t\right), \\
&\alpha_0, \alpha_1\left(s\right), \alpha_2\left(s\right), \alpha_3\left(s\right), \ \ldots \text{ given.}
\end{aligned}$$

In this formulation, it looks different from (I.2) because it directly includes the solution from the second phase of the game, but in fact, it is the same problem. The second constraint describes the law of motion for the experimentation rates that will be followed by the entrepreneur at the phase two, and $\alpha_1\left(s\right)$, $\alpha_2\left(s\right)$, $\alpha_3\left(s\right)$, and so on, are the belief targets that will be reached along the basic equilibrium experimentation path depending on the share, s, if the entrepreneur follows the experimentation path produced at the second phase of the game. In the basic case, there is no way to change these belief targets directly as they depend of the share.

Now, consider the phase one problem for the audited reporting case:

$$\begin{aligned}
&\max_{s} \int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \left[\alpha_t \epsilon_t R - f\left(\epsilon_t\right) - \epsilon_t c\right] \,\mathrm{d}t \\
\text{subject to: } &\int_0^\infty e^{-tr-\int_0^t \alpha_\tau \epsilon_\tau \,\mathrm{d}\tau} \alpha_t \epsilon_t \left(1-s\right) R \,\mathrm{d}t = \int_0^\infty e^{-tr} \epsilon_t c \,\mathrm{d}t, \\
&\dot{\epsilon}_t = \alpha_t \frac{f'\left(\epsilon_t\right)\epsilon_t - f\left(\epsilon_t\right)}{f''\left(\epsilon_t\right)} - r\frac{\alpha_t s R - f'\left(\epsilon_t\right) - c}{f''\left(\epsilon_t\right)}, \\
&\dot{\alpha}_t = -\alpha_t \epsilon_t \left(1-\alpha_t\right), \\
&\alpha_0, \alpha_1^*, \alpha_2^*, \alpha_3^*, \ \ldots \text{ given.}
\end{aligned}$$

It is exactly the same problem as in the one right above it, with the same laws of motion, and the same budget constraint; the only difference is the belief targets. In this problem, the belief targets are controlled directly and they coincide with the first best beliefs.

In this sense, the constraints in the second problem are more relaxed as the belief targets in the basic problem are deterministic and cannot be manually controlled. Having the manual control over the belief targets and knowing the first best belief levels, it is obvious that manually setting the belief targets at the first best levels will bring the value of the objective function closer to the first best value. Recall the the objective function for the first best problem and for the two problems above are exactly the same. It is possible that there exists a better solution, but it is definitely better to be able to control the belief levels manually and set the closer to the first best levels than to be unable to control them at all and have them constrained by the parameters of the model.

To sum up, by committing to the truthful audited disclosure of the aggregate experimentation rates at the efficient levels, the entrepreneur ensures that the path for the posterior beliefs is closer to the efficient path than it is without the commitment. Truthful disclosure is Pareto enhancing if the costs of disclosure are ignored.

Chapter II

Appendices

II.A. The Generalized Control Problem

This is the generalized problem of finding the optimal experimentation path given paths $(\gamma_t, t \geqslant 0)$ and $(s_t, t \geqslant 0)$. I begin by formulating the problem from the perspectives of Pontryagin's Maximum Principle:

$$J = \max_{(\epsilon_t, t \geqslant 0)} \left[\int_0^\infty e^{-rt - \int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau} \left(\alpha_t \epsilon_t \left(s_t R + E \right) - f\left(\epsilon_t\right) - \epsilon_t c + \gamma_t c \right) \, \mathrm{d}t \right]$$

$$\begin{aligned} \text{subject to } \dot{\alpha}_t &= -\alpha_t \epsilon_t \left(1 - \alpha_t\right) \\ s_t, \gamma_t &\text{ given } \forall t, \\ \text{and } \alpha_0, \alpha_\infty &\text{ given.} \end{aligned}$$

The main assumption is that α_∞ is finite.

Observe that

$$M_t \equiv e^{-\int_0^t \alpha_\tau \epsilon_\tau \, \mathrm{d}\tau},$$

so I can define

$$\dot{M} = -\alpha_t \epsilon_t M_t.$$

Rewrite the problem:

$$J = \max_{(\epsilon_t, t \geqslant 0)} \left[\int_0^\infty e^{-rt} M_t \left(\alpha_t \epsilon_t \left(s_t R + E \right) - f\left(\epsilon_t \right) - \epsilon_t c + \gamma_t c \right) \, \mathrm{d}t \right]$$

$$\begin{aligned} \text{subject to } & \dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t \right), \\ & \dot{M}_t = -\alpha_t \epsilon_t M_t \\ \text{and } & \alpha_0, \alpha_\infty \text{ given,} \\ & M_0 = 1. \end{aligned} \tag{II.a}$$

Assign Lagrange multipliers λ_t and μ_t to the first and the second constraints, respectively, assume that M_∞ is finite, and rewrite the problem in the Lagrangian form:

$$\begin{aligned} \mathcal{L} = & \int_0^\infty e^{-rt} M_t \left(\alpha_t \epsilon_t \left(s_t R + E \right) - f\left(\epsilon_t \right) - \epsilon_t c + \gamma_t c \right) \, \mathrm{d}t \\ & + \int_0^\infty \left[\lambda_t \left(-\alpha_t \epsilon_t \left(1 - \alpha_t \right) - \dot{\alpha}_t \right) + \mu_t \left(-\alpha_t \epsilon_t M_t - \dot{M}_t \right) \right] \, \mathrm{d}t \\ & \alpha_0, \alpha_\infty \text{ given,} \\ & M_0 = 1. \end{aligned}$$

Integrate by parts $\lambda_t \dot{\alpha}_t$ and $\mu_t \dot{M}_t$ and rewrite the problem:

$$\mathcal{L} = \int_0^\infty e^{-rt} M_t \left(\alpha_t \epsilon_t \left(s_t R + E \right) - f\left(\epsilon_t \right) - \epsilon_t c + \gamma_t c \right) \, \mathrm{d}t$$

$$- \int_0^\infty \left[\lambda_t \alpha_t \epsilon_t \left(1 - \alpha_t \right) - \mu_t \alpha_t \epsilon_t M_t + \dot{\lambda}_t \alpha_t + \dot{\mu}_t M_t \right] \, \mathrm{d}t - \left[\lambda_\infty \alpha_\infty - \lambda_0 \alpha_0 \right] - \left[\mu_\infty M_\infty - \mu_0 \right].$$

Define Hamiltonian function

$$\mathcal{H}_t = e^{-rt} M_t \left(\alpha_t \epsilon_t \left(s_t R + E \right) - f\left(\epsilon_t \right) - \epsilon_t c + \gamma_t c \right) - \lambda_t \alpha_t \epsilon_t \left(1 - \alpha_t \right) - \mu_t \alpha_t \epsilon_t M_t.$$

The necessary conditions are

$$\begin{aligned} \frac{\partial \mathcal{H}_t}{\partial \epsilon_t} &= e^{-rt} M_t \left(\alpha_t \left(s_t R + E \right) - f'\left(\epsilon_t \right) - c \right) - \lambda_t \alpha_t \left(1 - \alpha_t \right) - \mu_t \alpha_t M_t = 0, \\ \dot{\lambda}_t = -\frac{\partial \mathcal{H}_t}{\partial \alpha_t} &= - \left[e^{-rt} M_t \epsilon_t \left(s_t R + E \right) - \lambda_t \left(\epsilon_t \left(1 - \alpha_t \right) - \alpha_t \epsilon_t \right) - \mu_t \epsilon_t M_t \right], \\ \dot{\mu}_t = -\frac{\partial \mathcal{H}_t}{\partial M_t} &= - \left[e^{-rt} \left(\alpha_t \epsilon_t \left(s_t R + E \right) - f\left(\epsilon_t \right) - \epsilon_t c + \gamma_t c \right) - \mu_t \alpha_t \epsilon_t \right]; \end{aligned}$$

laws of motion of the state variables α_t and M_t,

$$\dot{\alpha}_t = -\alpha_t \epsilon_t \left(1 - \alpha_t\right) \text{ and}$$
$$\dot{M}_t = -\alpha_t \epsilon_t M_t;$$

and the boundary conditions of the state variables.

Define

$$\lambda_t \equiv e^{-rt}\bar{\lambda}_t, \qquad \text{so} \qquad \dot{\lambda}_t = -re^{-rt}\bar{\lambda}_t + e^{-rt}\dot{\bar{\lambda}}_t,$$
$$\mu_t \equiv e^{-rt}\bar{\mu}_t, \qquad \text{so} \qquad \dot{\mu}_t = -re^{-rt}\bar{\mu}_t + e^{-rt}\dot{\bar{\mu}}_t.$$

Using these new costate variables, rewrite the first three necessary conditions:

$$M_t \left(\alpha_t \left(s_t R + E\right) - f'\left(\epsilon_t\right) - c\right) - \bar{\lambda}_t \alpha_t \left(1 - \alpha_t\right) - \bar{\mu}_t \alpha_t M_t = 0,$$
$$-\dot{\bar{\lambda}}_t + \bar{\lambda}_t \epsilon_t \left(1 - 2\alpha_t\right) = M_t \epsilon_t \left(s_t R + E\right) - \bar{\mu}_t \epsilon_t M_t - r\bar{\lambda}_t,$$
$$-\dot{\bar{\mu}}_t + \bar{\mu}_t \alpha_t \epsilon_t = \alpha_t \epsilon_t \left(s_t R + E\right) - f\left(\epsilon_t\right) - \epsilon_t c + \gamma_t c - r\bar{\mu}_t.$$

Differentiate the first condition with respect to time:

$$\dot{M}_t \left(\alpha_t \left(s_t R + E\right) - f'\left(\epsilon_t\right) - c\right) + M_t \left(\dot{\alpha}_t \left(s_t R + E\right) + \alpha_t \dot{s}_t R - f''\left(\epsilon_t\right) \dot{\epsilon}_t\right)$$
$$+ \alpha_t \left(1 - \alpha_t\right) \left[-\dot{\bar{\lambda}}_t + \bar{\lambda}_t \epsilon_t \left(1 - 2\alpha_t\right)\right] - \dot{\bar{\mu}}_t \alpha_t M_t - \bar{\mu}_t \left[\dot{\alpha}_t M_t + \alpha_t \dot{M}_t\right] = 0,$$

plug the second condition into it:

$$\dot{M}_t \left(\alpha_t \left(s_t R + E\right) - f'\left(\epsilon_t\right) - c\right) + M_t \left(\dot{\alpha}_t \left(s_t R + E\right) + \alpha_t \dot{s}_t R - f''\left(\epsilon_t\right) \dot{\epsilon}_t\right)$$
$$+ \alpha_t \left(1 - \alpha_t\right) \left[M_t \epsilon_t \left(s_t R + E\right) - \bar{\mu}_t \epsilon_t M_t - r\bar{\lambda}_t\right] - \dot{\bar{\mu}}_t \alpha_t M_t - \bar{\mu}_t \left[\dot{\alpha}_t M_t + \alpha_t \dot{M}_t\right] = 0,$$

and simplify

$$\dot{M}_t \left(\alpha_t \left(s_t R + E\right) - f'\left(\epsilon_t\right) - c\right) + M_t \left(\alpha_t \dot{s}_t R - f''\left(\epsilon_t\right) \dot{\epsilon}_t\right)$$
$$- r\bar{\lambda}_t \alpha_t \left(1 - \alpha_t\right) + \alpha_t M_t \left[-\dot{\bar{\mu}}_t + \bar{\mu}_t \alpha_t \epsilon_t\right] = 0.$$

Plug the third necessary condition into it:

$$\dot{M}_t \left(\alpha_t \left(s_t R + E\right) - f'\left(\epsilon_t\right) - c\right) + M_t \left(\alpha_t \dot{s}_t R - f''\left(\epsilon_t\right) \dot{\epsilon}_t\right)$$

$$-r\bar{\lambda}_t\alpha_t\left(1-\alpha_t\right)+\alpha_t M_t\left[\alpha_t\epsilon_t\left(s_tR+E\right)-f\left(\epsilon_t\right)-\epsilon_t c+\gamma_t c-r\bar{\mu}_t\right]=0.$$

Simplify:

$$\alpha_t M_t\left[f'\left(\epsilon_t\right)\epsilon_t-f\left(\epsilon_t\right)+\gamma_t c\right]+M_t\left(\alpha_t\dot{s}_tR-f''\left(\epsilon_t\right)\dot{\epsilon}_t\right)$$
$$-r\left[\bar{\lambda}_t\alpha_t\left(1-\alpha_t\right)+\bar{\mu}_t\alpha_t M_t\right]=0.$$

Finally, use the first necessary condition in the last term to produce:

$$r\left[\alpha_t\left(s_tR+E\right)-f'\left(\epsilon_t\right)-c\right]=\alpha_t\left[f'\left(\epsilon_t\right)\epsilon_t-f\left(\epsilon_t\right)+\gamma_t c\right]+\alpha_t\dot{s}_tR-f''\left(\epsilon_t\right)\dot{\epsilon}_t. \tag{II.b}$$

Notice that M_t was canceled out. Given that $M_t > 0$ for any t, this can be done without any problems.

Therefore, the solution to the generalized control problem is represented by the system of first order ordinary differential equations,

$$\begin{cases}\dot{\epsilon}_t=\alpha_t\dfrac{\dot{s}_tR+f'\left(\epsilon_t\right)\epsilon_t-f\left(\epsilon_t\right)+\gamma_t c}{f''\left(\epsilon_t\right)}-r\dfrac{\alpha_t\left(s_tR+E\right)-f'\left(\epsilon_t\right)-c}{f''\left(\epsilon_t\right)},\\ \dot{\alpha}_t=-\alpha_t\epsilon_t\left(1-\alpha_t\right),\end{cases}$$

and boundary conditions,

$$\alpha_0,\ \alpha_\infty \text{ given.}$$

Recall that $f''(x) > 0$ for all x and that $f(\cdot)$ is twice continuously differentiable. It means that the right hand sides of the equations in the system above are continuously differentiable in ϵ and continuous in α for $\alpha \in (0,1)$. Assume that s_t and γ_t are continuously differentiable as well. Given that α_0 is known, it is enough to assume that $\epsilon_\infty = 0$ (instead of α_∞ given) to have the uniques solution according to the generalized version of the Picard–Lindelöf theorem.

II.B. The Efficient Solution

II.B.1. Existence and Uniqueness

The problem of finding the efficient experimentation path is:

$$\max_{(\epsilon_t, t\geqslant 0)}\int_0^\infty e^{-tr}M_t\left[\alpha_t\epsilon_t\left(R+E+I\right)-f\left(\epsilon_t\right)-\epsilon_t c\right]\,\mathrm{d}t$$

$$
\begin{aligned}
\text{subject to: } & \dot{\alpha}_t = -\alpha_t \epsilon_t (1-\alpha_t), \\
& \dot{M}_t = -\alpha_t \epsilon_t M_t, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0,1], \\
& M_0 = 1, \\
& \epsilon_t \geqslant 0, \forall t \geqslant 0.
\end{aligned}
$$

To make it look like the generalized problem, solved in Appendix II.A, I define, for all t:

$$
\begin{aligned}
\gamma_t &\equiv 0, \\
s_t &\equiv \frac{R+I}{R}.
\end{aligned}
$$

Then I know that the solution to the problem is unique and it follows the system of differential equations defined by (II.b) and the law of motion for the posterior belief α_t,

$$
\begin{cases}
\dot{\epsilon}_t = \alpha_t \dfrac{f'(\epsilon_t)\epsilon_t - f(\epsilon_t)}{f''(\epsilon_t)} - r\dfrac{\alpha_t (R+E+I) - f'(\epsilon_t) - c}{f''(\epsilon_t)}, \\
\dot{\alpha}_t = -\alpha_t \epsilon_t (1-\alpha_t).
\end{cases}
$$

Notice that nothing in the system depends on time explicitly and that α_t strictly decreases in time for all strictly positive ϵ_t. It means that α_t uniquely characterizes the state. I can rewrite this system in terms of the policy function, assuming that $\epsilon_t = \epsilon(\alpha_t)$:

$$
\begin{aligned}
& r\left[\alpha \bar{R} - f'(\epsilon^*(\alpha)) - c\right] \\
&= \alpha \left[f'(\epsilon^*(\alpha))\epsilon^*(\alpha) - f(\epsilon^*(\alpha)) + (1-\alpha) f''(\epsilon^*(\alpha))\epsilon^*(\alpha)\epsilon^{*\prime}(\alpha)\right],
\end{aligned} \tag{II.c}
$$

where $\bar{R} = R + E + I$, and the boundary condition,

$$
\epsilon^*(\underline{\alpha}) = 0.
$$

To produce the boundary condition, observe that there exists the lowest belief level of α such that it does not make sense to continue with experiments when it is reached. Suppose that $\underline{\alpha} = \frac{c}{\bar{R}}$. Then

$$
\underline{\alpha}\epsilon\bar{R} - f(\epsilon) - \epsilon c = \frac{c}{\bar{R}}\epsilon\bar{R} - f(\epsilon) - \epsilon c = -f(\epsilon) \leqslant 0,
$$

which means that the immediate benefits of experimenting are negative for any positive experimentation rate. Posterior belief α can only decrease in time, so if I still decide to experiment ignoring the loss, then in the future I will reach $\alpha < \underline{\alpha}$. My immediate benefits from experimenting will be

$$\alpha\epsilon\bar{R} - f(\epsilon) - \epsilon c < \underline{\alpha}\epsilon\bar{R} - f(\epsilon) - \epsilon c = -f(\epsilon) \leqslant 0.$$

Therefore, if the belief level of $\underline{\alpha}$ is reached then there is no reason to carry on with experiments, because it will only cause losses. Notice that it is not the case if $\alpha > \frac{c}{\bar{R}}$. Suppose that $\alpha = \underline{\alpha} + \delta$, where $\delta > 0$ is very small. Then

$$\alpha\epsilon\bar{R} - f(\epsilon) - \epsilon c = (\underline{\alpha} + \delta)\,\epsilon\bar{R} - f(\epsilon) - \epsilon c = \delta\epsilon\bar{R} - f(\epsilon),$$

and it is always possible to find ϵ to keep this expression positive. Therefore, the boundary condition exists and it is

$$\epsilon^*(\underline{\alpha}) = \epsilon^*\left(\frac{c}{\bar{R}}\right) = 0.$$

There is only one solution to the social planner's problem in terms of policy functions.

II.B.2. Properties of the Efficient Experimentation Path

The important properties of the efficient experimentation path are:

Staticity at the top : the experimentation rate for sure projects ($\alpha = 1$) is stationary. For belief level $\alpha = 1$, differential equation (II.c) degenerates to equation

$$r\left[\bar{R} - f'(\epsilon^*(1)) - c\right] = f'(\epsilon^*(1))\,\epsilon^*(1) - f(\epsilon^*(1)).$$

If $\alpha_0 = 1$, then for all t, $\alpha_t = 1$ as well:

$$\alpha_t = \frac{\alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}} = \frac{e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}}{e^{-\int_0^t \epsilon_\tau \, \mathrm{d}\tau}} = 1.$$

In this case, efficient experimentation level $\epsilon^*(1)$ is stationary, monotone, and strictly positive provided that $\bar{R} > c$ (so that $\frac{c}{\bar{R}} < 1$). Hence, sure projects must be worked on at the constant efficient experimentation rate of $\epsilon^*(1)$ until success happens.

Experimentation rate strictly decreases in time : for $\alpha \in \left(\frac{c}{\bar{R}}, 1\right)$, the experimentation rate *increases* in α, but since the posterior belief, α, strictly decreases over time given no success[1], the experimentation rate decreases in time. If $\epsilon^{*\prime}(\alpha) > 0$, it means that

$$\dot{\epsilon^*}(\alpha) = \dot{\alpha}\epsilon^{*\prime}(\alpha) = -\alpha\epsilon^*(\alpha)(1-\alpha)\epsilon^{*\prime}(\alpha) < 0.$$

Imagine that, on the contrary, that is not the case. Solutions to the first order differential equations are continuously differentiable. Since $\epsilon^*\left(\frac{c}{\bar{R}}\right) = 0$ and $\epsilon^*(1) > 0$, given $\frac{c}{\bar{R}} < 1$, experimentation rate $\epsilon^*(\alpha)$ must continuously increase somewhere for α between $\frac{c}{\bar{R}}$ and 1. So, for it to ever *decrease*, function $\epsilon^*(\alpha)$ must have at least one extreme point. Suppose there is such a point. Pick some $\hat{\alpha}$, differentiate both sides of (II.c) with respect to α and assume that $\epsilon^{*\prime}(\hat{\alpha}) = 0$:

$$r\bar{R} = f'(\epsilon^*(\hat{\alpha}))\epsilon^*(\hat{\alpha}) - f(\epsilon^*(\hat{\alpha})) + \hat{\alpha}(1-\hat{\alpha})f''(\epsilon^*(\hat{\alpha}))\epsilon^*(\hat{\alpha})\epsilon^{*\prime\prime}(\hat{\alpha}).$$

Next, plug $\epsilon^{*\prime}(\hat{\alpha}) = 0$ into (II.c) directly:

$$r\left[\hat{\alpha}\bar{R} - f'(\epsilon^*(\hat{\alpha})) - c\right] = \hat{\alpha}\left[f'(\epsilon^*(\hat{\alpha}))\epsilon^*(\hat{\alpha}) - f(\epsilon^*(\hat{\alpha}))\right].$$

Combine these two conditions together:

$$r\bar{R} = \frac{r\left[\hat{\alpha}\bar{R} - f'(\epsilon^*(\hat{\alpha})) - c\right]}{\hat{\alpha}} + \hat{\alpha}(1-\hat{\alpha})f''(\epsilon^*(\hat{\alpha}))\epsilon^*(\hat{\alpha})\epsilon^{*\prime\prime}(\hat{\alpha}),$$

and simplify:

$$r\frac{f'(\epsilon^*(\hat{\alpha})) + c}{\hat{\alpha}} = \hat{\alpha}(1-\hat{\alpha})f''(\epsilon^*(\hat{\alpha}))\epsilon^*(\hat{\alpha})\epsilon^{*\prime\prime}(\hat{\alpha}).$$

Given that $\hat{\alpha} \in \left(\frac{c}{\bar{R}}, 1\right)$, $\epsilon^* \geqslant 0$, $f'(\cdot) \geqslant 0$, and $f''(\cdot) > 0$, it must be the case that

$$\epsilon^{*\prime\prime}(\hat{\alpha}) > 0$$

when $\epsilon^{*\prime}(\hat{\alpha}) = 0$. It means that *all* the extreme points that can be found along the efficient experimentation path must be local minima. It is only possible if $\epsilon^*(\alpha)$ first strictly decreases and then strictly increases on $\left(\frac{c}{\bar{R}}, 1\right)$, but this is impossible: $\epsilon^*\left(\frac{c}{\bar{R}}\right) = 0$, and experimentation rates cannot be negative. So there are no extreme points and function $\epsilon^*(\alpha)$ strictly increases in α and strictly decreases over time.

[1] Recall that $\frac{\mathrm{d}\alpha}{\mathrm{d}t} \equiv \dot{\alpha} = -\alpha\epsilon(1-\alpha) < 0$.

Experimenting never stops : despite that experimentation rate strictly decreases over time, experiments never stop. To show that experiments never stop, I demonstrate that for $\alpha \in \left[\frac{c}{\bar{R}}, 1\right]$ efficient policy function $\epsilon^*(\alpha)$ is everywhere below critical function $\bar{\epsilon}(\alpha)$ from Appendix C and that $\epsilon^*(\alpha) > 0$ for all $\alpha \in \left(\frac{c}{\bar{R}}, 1\right]$:

- function $\epsilon^*(\alpha)$ is a solution to an ordinary differential equation, it is continuous;
- $\epsilon^*(\alpha) > 0$ on $\left(\frac{c}{\bar{R}}, 1\right]$ as it is strictly increasing from $\epsilon\left(\frac{c}{\bar{R}}\right) = 0$ to $\epsilon^*(1) < \infty$, so it is bounded.

The only condition that must be verified is $\epsilon^{*\prime}(\underline{\alpha}) < \infty$. Express $\epsilon^{*\prime}(\underline{\alpha})$ from (II.c) and take limits:

$$\begin{aligned}
\epsilon^{*\prime}\left(\frac{c}{\bar{R}}\right) &= \lim_{\alpha \to \frac{c}{\bar{R}}} \frac{r\left[\alpha\bar{R} - f'(\epsilon^*(\alpha)) - c\right] - \alpha f'(\epsilon^*(\alpha))\,\epsilon^*(\alpha) + \alpha f(\epsilon^*(\alpha))}{\alpha(1-\alpha) f''(\epsilon^*(\alpha))\,\epsilon^*(\alpha)} \\
&= \frac{r\bar{R}^2\left[\frac{\bar{R}}{\epsilon^{*\prime}\left(\frac{c}{\bar{R}}\right)} - f''(0)\right]}{c\left(\bar{R}-c\right) f''(0)}.
\end{aligned}$$

Assuming $\epsilon^{*\prime}\left(\frac{c}{\bar{R}}\right) = \infty$ will be inconsistent with this expression, so it must be the case that $\epsilon^{*\prime}\left(\frac{c}{\bar{R}}\right) < \infty$.

Therefore, the efficient experimentation rate strictly decreases indefinitely over time until the project is a success or forever. This confirms the assumption that for all $t \geqslant 0$, $\epsilon_t^* > 0$, thus α_t uniquely characterizes the state.

II.C. The Equilibrium

II.C.1. Binding Budget Constraint

The original problem with the binding budget constraint is

$$\begin{aligned}
&\max_{(\epsilon_t, t \geqslant 0)} \left[\int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t (s_t R + E) - f(\epsilon_t) - \epsilon_t c + \gamma_t c\right] \mathrm{d}t\right] \\
\text{subject to: } &\int_0^\infty e^{-rt} M_t (\epsilon_t c - \gamma_t c)\, \mathrm{d}t = 0, \\
&\dot{\alpha}_t = -\alpha_t \epsilon_t (1-\alpha_t), \\
&\dot{M}_t = -\alpha_t \epsilon_t M_t,
\end{aligned}$$

$$(s_t, t \geqslant 0), \ (\gamma_t, t \geqslant 0) \ \text{given,}$$

$$\alpha_0 \text{ given, } \alpha_0 \in [0,1],$$

$$M_0 = 1,$$

$$\epsilon_t \geqslant 0, \forall t \geqslant 0.$$

Suppose the Lagrangian multiplier associated with the first constraint is ν. Then the problem can be written as:

$$\max_{(\epsilon_t, t \geqslant 0), \nu} \left[\int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t (s_t R + E) - f(\epsilon_t) - \epsilon_t c + \gamma_t c \right] \mathrm{d}t + \nu \int_0^\infty e^{-rt} M_t (\gamma_t c - \epsilon_t c) \ \mathrm{d}t \right]$$

$$\text{subject to: } \dot{\alpha}_t = -\alpha_t \epsilon_t (1 - \alpha_t),$$

$$\dot{M}_t = -\alpha_t \epsilon_t M_t,$$

$$(s_t, t \geqslant 0), \ (\gamma_t, t \geqslant 0) \ \text{given,}$$

$$\alpha_0 \text{ given, } \alpha_0 \in [0,1],$$

$$M_0 = 1,$$

$$\epsilon_t \geqslant 0, \forall t \geqslant 0.$$

Simplify:

$$\max_{(\epsilon_t, t \geqslant 0), \nu} \left[\int_0^\infty e^{-tr} M_t \left[\alpha_t \epsilon_t (s_t R + E) - f(\epsilon_t) - \epsilon_t (1 + \nu) c + \gamma_t (1 + \nu) c \right] \mathrm{d}t \right]$$

$$\text{subject to: } \dot{\alpha}_t = -\alpha_t \epsilon_t (1 - \alpha_t),$$

$$\dot{M}_t = -\alpha_t \epsilon_t M_t,$$

$$(s_t, t \geqslant 0), \ (\gamma_t, t \geqslant 0) \ \text{given,}$$

$$\alpha_0 \text{ given, } \alpha_0 \in [0,1],$$

$$M_0 = 1,$$

$$\epsilon_t \geqslant 0, \forall t \geqslant 0.$$

Thus the problem with the binding budget constraint is equivalent to the problem without any budget constraint but with a higher marginal monetary cost of experimentation, c. The more stiff the budget constraint is, the costlier it is to conduct experiments.

II.C.2. Incentives to Experiment and Conditional Payments

First, I argue that the budget constraint binds. Suppose it does not. Recall that the incentives to experiment at stage two depend on

$$\omega_t = R\left(\dot{s}_t - rs_t\right) + \gamma_t c.$$

If the budget constraint does not bind, it is possible to decrease γ_t for every period by multiplying it by some number $g < 1$, which is very close to one. Then it will be possible to induce the same experimentation path as before the change by setting

$$\begin{aligned}\omega_t &= R\left(\dot{s}_t - rs_t\right) + \gamma_t c\left(1-g\right) + g\gamma_t c \\ &= R\left[\dot{s}_t - r\left(s_t - \left(1-g\right)\frac{\gamma_t c}{rR}\right)\right] + g\gamma_t c.\end{aligned}$$

Since the objective function at phase one depends only on the experimentation path and does not depend on shares or funding rates directly, this change will not make any difference for the entrepreneur.

Consider then the investor's participation constraint from stage one:

$$\int_0^\infty e^{-rt-\int_0^t \hat{\alpha}_\tau\hat{\epsilon}_\tau\,\mathrm{d}\tau}\left[\hat{\alpha}_t\hat{\epsilon}_t\left[\left(1-s_t\right)R + I\right] - \gamma_t c\right]\,\mathrm{d}t = 0.$$

The effects of changing the funding rate by multiplying it by $g < 1$ everywhere while adjusting the shares to compensate for this change will results in the new condition,

$$\begin{aligned}&\int_0^\infty e^{-rt-\int_0^t \hat{\alpha}_\tau\hat{\epsilon}_\tau\,\mathrm{d}\tau}\left[\hat{\alpha}_t\hat{\epsilon}_t\left[\left(1-s_t+\left(1-g\right)\frac{\gamma_t c}{rR}\right)R + I\right] - g\gamma_t c\right]\,\mathrm{d}t \\ =&\int_0^\infty e^{-rt-\int_0^t \hat{\alpha}_\tau\hat{\epsilon}_\tau\,\mathrm{d}\tau}\left[\hat{\alpha}_t\hat{\epsilon}_t\left[\left(1-s_t\right)R + I\right) - \gamma_t c + \gamma_t c\left(1-g\right)\frac{\hat{\alpha}_t\hat{\epsilon}_t + r}{r}\right]\,\mathrm{d}t > 0.\end{aligned}$$

The investor's participation constraint will no longer bind. However, it was established that the investor's participation constraint must bind. The contradiction has been reached. Therefore, the budget constraint must bind as well.

Second, I argue that at the second stage the budget constraint does not bind. Suppose it does. Then the experimentation rates are determined by

$$r\left[\hat{\alpha}_t\left(s_t R + E\right) - f'\left(\hat{\epsilon}_t\right) - c\left(1+\nu\right)\right]$$

$$= \hat{\alpha}_t \left[f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right) + \gamma_t c\left(1+\nu\right)\right] + \hat{\alpha}_t \dot{s}_t R - f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t,$$

where ν is the Lagrangian multiplier from Appendix II.C.1. Alternatively,

$$\begin{aligned} &r\left[\hat{\alpha}_t\left(\left(s_t - \frac{\hat{\alpha}_t\gamma_t + r}{r\hat{\alpha}_t R}c\nu\right)R + E\right) - f'\left(\hat{\epsilon}_t\right) - c\right] \\ &= \hat{\alpha}_t \dot{s}_t R - f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t + \hat{\alpha}_t\left[f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right) + \gamma_t c\right]. \end{aligned}$$

Therefore, for every belief level, the shares can be decreased by $\frac{\hat{\alpha}_t\gamma_t+r}{r\hat{\alpha}_t R}c\nu$, which will not affect the experimentation path and thus the objective function at stage one. However, if every share can be decreased, then the investor's participation constraint will no longer be binding as his share will increase for every belief level, other things equal. The investor's participation constraint must be binding in equilibrium. This is a contradiction. Therefore, the budget constraint at phase two is not restrictive.

II.C.3. Phase One Problem Statement

The problem the entrepreneur is facing at the beginning of phase one is

$$\begin{aligned} \max_{((s_t,\gamma_t),t\geqslant 0)} &\left[\int_0^\infty e^{-tr} M_t\left[\hat{\alpha}_t\hat{\epsilon}_t\left(s_t R + E\right) - f\left(\hat{\epsilon}_t\right) - \hat{\epsilon}_t c + \gamma_t c\right]\,\mathrm{d}t\right] \\ \text{subject to: } &\int_0^\infty e^{-rt} M_t\left[\hat{\alpha}_t\hat{\epsilon}_t\left[\left(1-s_t\right)R + I\right] - \gamma_t c\right]\,\mathrm{d}t = 0, \\ &\int_0^\infty e^{-rt} M_t\left[\gamma_t c - \hat{\epsilon}_t c\right]\,\mathrm{d}t = 0, \\ &\dot{\hat{\alpha}}_t = -\hat{\alpha}_t\hat{\epsilon}_t\left(1-\hat{\alpha}_t\right), \\ &\dot{M}_t = -\hat{\alpha}_t\hat{\epsilon}_t M_t, \\ &\alpha_0 \text{ given, } \alpha_0 \in [0,1], \\ &M_0 = 1. \end{aligned}$$

Experimentation path $\hat{\epsilon}_t$ depends on the funding rates and the shares.

Assign multiplier λ to the first constraint and multiplier μ to the second constraint. Rewrite the problem in the Lagrangian form:

$$\max_{((s_t,\gamma_t),t\geqslant 0),\lambda,\mu} \left[\int_0^\infty e^{-tr} M_t\left[\hat{\alpha}_t\hat{\epsilon}_t\left(s_t R + E\right) - f\left(\hat{\epsilon}_t\right) - \hat{\epsilon}_t c + \gamma_t c\right]\,\mathrm{d}t\right.$$

$$
\begin{aligned}
&+ \lambda \int_0^\infty e^{-rt} M_t \left[\hat{\alpha}_t \hat{\epsilon}_t \left[\left(1 - s_t\right) R + I\right] - \gamma_t c\right] \mathrm{d}t \\
&+ \mu \int_0^\infty e^{-rt} M_t \left[\gamma_t c - \hat{\epsilon}_t c\right] \mathrm{d}t \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to: } & \dot{\hat{\alpha}}_t = -\hat{\alpha}_t \hat{\epsilon}_t \left(1 - \hat{\alpha}_t\right), \\
& \dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\
& M_0 = 1.
\end{aligned}
$$

Simplify:

$$
\begin{aligned}
\max_{((s_t, \gamma_t), t \geqslant 0), \lambda, \mu} & \left[\int_0^\infty e^{-tr} M_t \left[\hat{\alpha}_t \hat{\epsilon}_t \left(s_t R + E\right) - f\left(\hat{\epsilon}_t\right) - \left(1 + \mu\right) \hat{\epsilon}_t c + \left(1 - \lambda + \mu\right) \gamma_t c\right.\right. \\
& \left.\left. + \lambda \hat{\alpha}_t \hat{\epsilon}_t \left(\left(1 - s_t\right) R + I\right)\right] \mathrm{d}t\right] \\
\text{subject to: } & \dot{\hat{\alpha}}_t = -\hat{\alpha}_t \hat{\epsilon}_t \left(1 - \hat{\alpha}_t\right), \\
& \dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t, \\
& \alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\
& M_0 = 1.
\end{aligned}
$$

Treat λ and μ as numbers for now. The behavior of the entrepreneur at phase two of the game is determined by differential equation (II.2). Written in terms of induced experimentation rates, it becomes

$$
r\left[\hat{\alpha}_t \left(s_t R + E\right) - f'\left(\hat{\epsilon}_t\right) - c\right] = \hat{\alpha}_t \dot{s}_t R - f''\left(\hat{\epsilon}_t\right) \dot{\hat{\epsilon}}_t + \hat{\alpha}_t \left[f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right) + \gamma_t c\right].
$$

This differential equation is hard to solve for $\hat{\epsilon}_t$, but it can be solved for s_t. Rearrange the terms:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(s_t R + E\right) = r\left(s_t R + E\right) - r \frac{f'\left(\hat{\epsilon}_t\right) + c}{\hat{\alpha}_t} + \frac{f''\left(\hat{\epsilon}_t\right) \dot{\hat{\epsilon}}_t}{\hat{\alpha}_t} - f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + f\left(\hat{\epsilon}_t\right) - \gamma_t c.
$$

This is a first order linear differential equation. The solution is

$$
s_t R + E = e^{rt} k + e^{rt} \int_0^t e^{-r\tau} \left[-r \frac{f'\left(\hat{\epsilon}_\tau\right) + c}{\hat{\alpha}_\tau} + \frac{f''\left(\hat{\epsilon}_\tau\right) \dot{\hat{\epsilon}}_\tau}{\hat{\alpha}_\tau} - f'\left(\hat{\epsilon}_\tau\right) \hat{\epsilon}_\tau + f\left(\hat{\epsilon}_\tau\right) - \gamma_\tau c\right] \mathrm{d}\tau,
$$

where k is some constant. Define

$$g_\tau \equiv -r\frac{f'(\hat{\epsilon}_\tau)+c}{\hat{\alpha}_\tau}+\frac{f''(\hat{\epsilon}_\tau)\,\dot{\hat{\epsilon}}_\tau}{\hat{\alpha}_\tau}-f'(\hat{\epsilon}_\tau)\,\hat{\epsilon}_\tau+f(\hat{\epsilon}_\tau)-\gamma_\tau c,$$

then

$$s_tR+E=e^{rt}k+e^{rt}\int_0^t e^{-r\tau}g_\tau\,\mathrm{d}\tau$$

Pick some time $T>t$ and express

$$\begin{aligned}
s_TR+E&=e^{rT}k+e^{rT}\int_0^T e^{-r\tau}g_\tau\,\mathrm{d}\tau\\
&=e^{rT}k+e^{rT}\int_0^t e^{-r\tau}g_\tau\,\mathrm{d}\tau+e^{rT}\int_t^T e^{-r\tau}g_\tau\,\mathrm{d}\tau\\
&=e^{r(T-t)}\left(s_tR+E\right)+e^{rT}\int_t^T e^{-r\tau}g_\tau\,\mathrm{d}\tau.
\end{aligned}$$

Rearrange the terms:

$$s_tR+E=e^{-r(T-t)}\left(s_TR+E\right)-e^{-r(T-t)}e^{rT}\int_t^T e^{-r\tau}g_\tau\,\mathrm{d}\tau.$$

In theory, the experiments can continue without bound if the project remains unsuccessful. Thus when $T\to\infty$, then, assuming that the share is bounded,

$$\begin{aligned}
s_tR+E&=-\int_t^\infty e^{-r(\tau-t)}g_\tau\,\mathrm{d}\tau\\
&=-\int_t^\infty e^{-r(\tau-t)}\left[-r\frac{f'(\hat{\epsilon}_\tau)+c}{\hat{\alpha}_\tau}+\frac{f''(\hat{\epsilon}_\tau)\,\dot{\hat{\epsilon}}_\tau}{\hat{\alpha}_\tau}-f'(\hat{\epsilon}_\tau)\,\hat{\epsilon}_\tau+f(\hat{\epsilon}_\tau)-\gamma_\tau c\right]\mathrm{d}\tau.
\end{aligned}$$

Notice that

$$\begin{aligned}
\int_t^\infty e^{-r(\tau-t)}\frac{f''(\hat{\epsilon}_\tau)\,\dot{\hat{\epsilon}}_\tau}{\hat{\alpha}_\tau}\,\mathrm{d}\tau&=\int_t^\infty\frac{e^{-r(\tau-t)}}{\hat{\alpha}_\tau}\frac{\mathrm{d}f'(\hat{\epsilon}_\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau\\
&=\left.\frac{e^{-r(\tau-t)}}{\hat{\alpha}_\tau}f'(\hat{\epsilon}_\tau)\right|_t^\infty-\int_t^\infty\left(-r\frac{e^{-r(\tau-t)}}{\hat{\alpha}_\tau}-\frac{e^{-r(\tau-t)}}{\hat{\alpha}_\tau^2}\dot{\hat{\alpha}}_\tau\right)f'(\hat{\epsilon}_\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau\\
&=-\frac{f'(\hat{\epsilon}_t)}{\hat{\alpha}_t}+\int_t^\infty e^{-r(\tau-t)}\left(r-\hat{\epsilon}_\tau\left(1-\hat{\alpha}_\tau\right)\right)\frac{f'(\hat{\epsilon}_\tau)}{\hat{\alpha}_\tau}\mathrm{d}\tau\,\mathrm{d}\tau.
\end{aligned}$$

Therefore,

$$s_tR+E=\frac{f'(\hat{\epsilon}_t)}{\hat{\alpha}_t}+\int_t^\infty e^{-r(\tau-t)}\left[\frac{f'(\hat{\epsilon}_\tau)\,\hat{\epsilon}_\tau+rc}{\hat{\alpha}_\tau}-f(\hat{\epsilon}_\tau)+\gamma_\tau c\right]\mathrm{d}\tau. \tag{II.d}$$

This expression characterizes the part of the surplus that the entrepreneur should receive if experiments succeed at time t to follow the experimentation path $(\hat{\epsilon}_t, t \geqslant 0)$ given funding path $(\gamma_t, t \geqslant 0)$. In other words, instead of controlling the shares and the funding rates that induce experimentation path $(\hat{\epsilon}_t, t \geqslant 0)$, the entrepreneur can control the experimentation path directly while setting the shares according to (II.d) each time period.

Rearrange the terms in the phase one problem:

$$\max_{((s_t,\gamma_t),t\geqslant 0),\lambda,\mu} \left[\int_0^\infty e^{-tr} M_t \left[\hat{\alpha}_t\hat{\epsilon}_t s_t R\left(1-\lambda\right) + \hat{\alpha}_t\hat{\epsilon}_t\left(\lambda R + E + \lambda I\right) - f\left(\hat{\epsilon}_t\right) - \left(1+\mu\right)\hat{\epsilon}_t c \right.\right.$$
$$\left.\left. + \left(1-\lambda+\mu\right)\gamma_t c\right] \mathrm{d}t\right]$$

subject to: $\dot{\hat{\alpha}}_t = -\hat{\alpha}_t\hat{\epsilon}_t\left(1-\hat{\alpha}_t\right),$

$\dot{M}_t = -\hat{\alpha}_t\hat{\epsilon}_t M_t,$

α_0 given, $\alpha_0 \in [0,1],$

$M_0 = 1.$

Express $s_t R$ from (II.d), insert it into the objective function and simplify:

$$\max_{((\hat{\epsilon}_t,\gamma_t),t\geqslant 0),\lambda,\mu} \int_0^\infty e^{-tr} M_t \left[\lambda\hat{\alpha}_t\hat{\epsilon}_t\left(R+E+I\right) + \left(1-\lambda\right) f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right)\right.$$
$$+ \left(1-\lambda+\mu\right)\gamma_t c - \left(1+\mu\right)\hat{\epsilon}_t c$$
$$\left. + \hat{\alpha}_t\hat{\epsilon}_t\left(1-\lambda\right)\int_t^\infty e^{-r(\tau-t)}\left(\frac{f'\left(\hat{\epsilon}_\tau\right)\hat{\epsilon}_\tau + rc}{\hat{\alpha}_\tau} - f\left(\hat{\epsilon}_\tau\right) + \gamma_\tau c\right)\mathrm{d}\tau\right]\mathrm{d}t$$

subject to: $\dot{\hat{\alpha}}_t = -\hat{\alpha}_t\hat{\epsilon}_t\left(1-\hat{\alpha}_t\right),$

$\dot{M}_t = -\hat{\alpha}_t\hat{\epsilon}_t M_t,$

α_0 given, $\alpha_0 \in [0,1],$

$M_0 = 1.$

Notice that one of the control variables got changed from the share path to the experimentation path.

Integrate by parts the last term in the objective function:

$$\int_0^\infty e^{-rt} M_t\hat{\alpha}_t\hat{\epsilon}_t \int_t^\infty e^{-r(\tau-t)}\left[\frac{f'\left(\hat{\epsilon}_\tau\right)\hat{\epsilon}_\tau + rc}{\hat{\alpha}_\tau} - f\left(\hat{\epsilon}_\tau\right) + \gamma_\tau c\right]\mathrm{d}\tau\,\mathrm{d}t$$

$$= \int_0^\infty e^{-\int_0^t \hat{\alpha}_\theta \hat{\epsilon}_\theta \, \mathrm{d}\theta} \hat{\alpha}_t \hat{\epsilon}_t \int_t^\infty e^{-r\tau} \left[\frac{f'(\hat{\epsilon}_\tau)\,\hat{\epsilon}_\tau + rc}{\hat{\alpha}_\tau} - f(\hat{\epsilon}_\tau) + \gamma_\tau c \right] \mathrm{d}\tau \, \mathrm{d}t$$

$$= -\int_0^\infty \frac{\mathrm{d}e^{-\int_0^t \hat{\alpha}_\theta \hat{\epsilon}_\theta \, \mathrm{d}\theta}}{\mathrm{d}t} \int_t^\infty e^{-r\tau} \left[\frac{f'(\hat{\epsilon}_\tau)\,\hat{\epsilon}_\tau + rc}{\hat{\alpha}_\tau} - f(\hat{\epsilon}_\tau) + \gamma_\tau c \right] \mathrm{d}\tau \, \mathrm{d}t$$

$$= \int_0^\infty e^{-rt} (1 - M_t) \left[\frac{f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t) + \gamma_t c \right] \mathrm{d}t.$$

Insert this expression back into the objective function to finally produce the phase one problem statement:

$$\max_{((\hat{\epsilon}_t, \gamma_t), t \geqslant 0), \lambda, \mu} \int_0^\infty e^{-tr} \Big[M_t \left(\lambda \hat{\alpha}_t \hat{\epsilon}_t (R + E + I) + (1-\lambda) f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t - f(\hat{\epsilon}_t) \right.$$

$$\left. + (1 - \lambda + \mu)\, \gamma_t c - (1 + \mu)\, \hat{\epsilon}_t c \right)$$

$$+ (1-\lambda)(1 - M_t) \left(\frac{f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t) + \gamma_t c \right) \Big] \mathrm{d}t$$

subject to: $\dot{\hat{\alpha}}_t = -\hat{\alpha}_t \hat{\epsilon}_t (1 - \hat{\alpha}_t)$,

$\dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t$,

α_0 given, $\alpha_0 \in [0, 1]$,

$M_0 = 1$.

This is the general statement of the phase one problems of the entrepreneur. I provide two solutions to it. The first solution is the "absolute" solution with unconditional payments. The second solution is for the situation when the conditional payments are exactly equal to the amount of funds the entrepreneur needs each period.

II.C.4. The Best Contract

First thing to notice about the phase one problem statement is that conditional payments γ_t do not affect the laws of motion of the state variables and enter the objective function as simple terms every period. Hence I can infer that one of the necessary conditions, for every period t, will be simply

$$e^{-rt} \left[M_t (1 - \lambda + \mu)\, c + (1 - \lambda)(1 - M_t)\, c \right] = 0,$$

or just

$$M_t \mu + (1 - \lambda) = 0.$$

Given that

$$\dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t$$

and multipliers stay constant, this condition can only be satisfied if $\hat{\epsilon}_t = 0$ at all times. This cannot be a solution. Otherwise, this condition will not be satisfied. However, there is an important inference to be made. Suppose that this condition is satisfied at time t. Then the entrepreneur will want infinite funding rate prior to t and no funding after that. It means that just by moving funding closer to time zero, the entrepreneur creates an opportunity to relax the budget constraint. Recall that the terms of the contract do not explicitly affect the objective function at phase one. What matters is the experimentation rate. There is a tradeoff between the experimentation rate and funding, which can be seen in (II.d). The economics behind it is simple: conditional payments create incentives to delay experiments because they are only received in the case of failure. They provide incentives to fail and must be countered by higher conditional rewards. Thus it is optimal to move all the funds to time zero.

Therefore, the best contract does not have conditional payments. The funding is either provided upfront or unconditionally over time. Unconditional funding means that the entrepreneur will receive funds independent of the future success of the project. This way, the entrepreneur treats the funding scheme as something given after the contract is signed. It is just her own money now, the only way they affect the experiments is by keeping the budget constraint binding.

Practically, it means that instead of the funding path, it is enough to have

$$P = \int_0^t e^{-rt} M_t \gamma_t c \, \mathrm{d}t,$$

where P is the present value of all the unconditional funds the entrepreneur will receive in the future, or just an upfront payment. The budget constraint becomes

$$P = \int_0^\infty e^{-rt} M_t \hat{\epsilon}_t c \, \mathrm{d}t,$$

and the investor's participation constraint becomes

$$\int_0^\infty e^{-rt-\int_0^t \hat{\alpha}_\tau \hat{\epsilon}_\tau \, \mathrm{d}\tau} \hat{\alpha}_t \hat{\epsilon}_t \left[(1 - s_t) R + I \right] \, \mathrm{d}t = P.$$

Using the expression for P from the budget constraint in the participation constraint and in the objective function, it is possible to simplify the problem and remove funding rate $(\gamma_t, t \geqslant 0)$ and multiplier μ from the set of control variables.

The next important observation is that given that there will be no conditional funding in the future, funding rate γ_t no longer influences the experimentation rate at the second phase of the game. Thus, (II.d) degenerates to

$$s_t R + E = \frac{f'(\hat{\epsilon}_t)}{\hat{\alpha}_t} + \int_t^{\infty} e^{-r(\tau - t)} \left[\frac{f'(\hat{\epsilon}_\tau)\,\hat{\epsilon}_\tau + rc}{\hat{\alpha}_\tau} - f(\hat{\epsilon}_\tau) \right] \mathrm{d}\tau.$$

Therefore, the problem of finding the optimal contract is just a problem of finding the optimal experimentation path:

$$\begin{aligned} J = \max_{(\hat{\epsilon}_t, t \geqslant 0), \lambda} \int_0^{\infty} e^{-tr} \Big[& M_t \left(\lambda \hat{\alpha}_t \hat{\epsilon}_t (R + E + I) + (1 - \lambda) f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t - f(\hat{\epsilon}_t) - \lambda \hat{\epsilon}_t c \right) \\ & + (1 - \lambda)(1 - M_t) \left(\frac{f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t) \right) \Big] \, \mathrm{d}t \end{aligned}$$

$$\begin{aligned} \text{subject to: } & \dot{\hat{\alpha}}_t = -\hat{\alpha}_t \hat{\epsilon}_t (1 - \hat{\alpha}_t), \\ & \dot{M}_t = -\hat{\alpha}_t \hat{\epsilon}_t M_t, \\ & \alpha_0 \text{ given}, \alpha_0 \in [0, 1], \\ & M_0 = 1. \end{aligned}$$

This is a dynamic control problem. It is best approached from the perspectives of Pontryagin's Maximum Principle.

Assign multipliers ν_t and ξ_t to the first and the second constraint from this problem, respectively. Assume that the end values for the state variables are known, and write the Lagrangian:

$$\begin{aligned} \mathcal{L} = \int_0^{\infty} \Big[& e^{-tr} M_t \left[\lambda \hat{\alpha}_t \hat{\epsilon}_t (R + E + I) + (1 - \lambda) f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t - f(\hat{\epsilon}_t) - \lambda \hat{\epsilon}_t c \right] \\ & + e^{-tr} (1 - \lambda)(1 - M_t) \left[\frac{f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t) \right] \\ & - \nu_t \left[\hat{\alpha}_t \hat{\epsilon}_t (1 - \hat{\alpha}_t) + \dot{\hat{\alpha}}_t \right] - \xi_t \left[\hat{\alpha}_t \hat{\epsilon}_t M_t + \dot{M}_t \right] \Big] \, \mathrm{d}t. \end{aligned}$$

Integrate by parts the terms containing $\nu_t \dot{\hat{\alpha}}_t$ and $\xi_t \dot{M}_t$:

$$\mathcal{L} = \int_0^{\infty} \Big[e^{-tr} M_t \left[\lambda \hat{\alpha}_t \hat{\epsilon}_t (R + E + I) + (1 - \lambda) f'(\hat{\epsilon}_t)\,\hat{\epsilon}_t - f(\hat{\epsilon}_t) - \lambda \hat{\epsilon}_t c \right]$$

$$+ e^{-tr}(1-\lambda)(1-M_t)\left[\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t)\right]$$
$$\left. - \nu_t\hat{\alpha}_t\hat{\epsilon}_t(1-\hat{\alpha}_t) - \xi_t\hat{\alpha}_t\hat{\epsilon}_t M_t + \dot{\nu}_t\hat{\alpha}_t + \dot{\xi}_t M_t\right] \mathrm{d}t + K,$$

where K is some constant term, which does not affect the maximization problem.

The Hamiltonian function for this problem is

$$\mathcal{H}_t = e^{-tr}M_t\left[\lambda\hat{\alpha}_t\hat{\epsilon}_t(R+E+I) + (1-\lambda)f'(\hat{\epsilon}_t)\hat{\epsilon}_t - f(\hat{\epsilon}_t) - \lambda\hat{\epsilon}_t c\right]$$
$$+ e^{-tr}(1-\lambda)(1-M_t)\left[\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t)\right]$$
$$- \nu_t\hat{\alpha}_t\hat{\epsilon}_t(1-\hat{\alpha}_t) - \xi_t\hat{\alpha}_t\hat{\epsilon}_t M_t.$$

The necessary conditions are:

$$\frac{\partial\mathcal{H}_t}{\partial\hat{\epsilon}_t} = e^{-tr}M_t\left[\lambda\hat{\alpha}_t(R+E+I) + (1-\lambda)f''(\hat{\epsilon}_t)\hat{\epsilon}_t - \lambda f'(\hat{\epsilon}_t) - \lambda c\right]$$
$$+ e^{-tr}(1-\lambda)(1-M_t)\left[\frac{f''(\hat{\epsilon}_t)\hat{\epsilon}_t + f'(\hat{\epsilon}_t)}{\hat{\alpha}_t} - f'(\hat{\epsilon}_t)\right]$$
$$- \nu_t\hat{\alpha}_t(1-\hat{\alpha}_t) - \xi_t\hat{\alpha}_t M_t = 0,$$

$$\dot{\nu}_t = -\frac{\partial\mathcal{H}_t}{\partial\hat{\alpha}_t} = -e^{-tr}M_t\lambda\hat{\epsilon}_t(R+E+I) + e^{-tr}(1-\lambda)(1-M_t)\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t^2}$$
$$+ \nu_t\hat{\epsilon}_t(1-2\hat{\alpha}_t) + \xi_t\hat{\epsilon}_t M_t,$$

$$\dot{\xi}_t = -\frac{\partial\mathcal{H}_t}{\partial M_t} = -e^{-tr}\left[\lambda\hat{\alpha}_t\hat{\epsilon}_t(R+E+I) + (1-\lambda)f'(\hat{\epsilon}_t)\hat{\epsilon}_t - f(\hat{\epsilon}_t) - \lambda\hat{\epsilon}_t c\right]$$
$$+ e^{-tr}(1-\lambda)\left[\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t)\right]$$
$$+ \xi_t\hat{\alpha}_t\hat{\epsilon}_t,$$

as well as laws of motion and the boundary conditions.

Define

$$\nu_t \equiv e^{-rt}\bar{\nu}_t, \qquad \Rightarrow \qquad \dot{\nu}_t = -re^{-rt}\bar{\nu}_t + e^{-rt}\dot{\bar{\nu}}_t,$$
$$\xi_t \equiv e^{-rt}\bar{\xi}_t, \qquad \Rightarrow \qquad \dot{\xi}_t = -re^{-rt}\bar{\xi}_t + e^{-rt}\dot{\bar{\xi}}_t.$$

Using these new costate variables, rewrite the necessary conditions:

$$\begin{aligned}
&M_t \left[\lambda \hat{\alpha}_t (R+E+I) + (1-\lambda) f''(\hat{\epsilon}_t)\hat{\epsilon}_t - \lambda f'(\hat{\epsilon}_t) - \lambda c\right] \\
&+ (1-\lambda)(1-M_t)\left[\frac{f''(\hat{\epsilon}_t)\hat{\epsilon}_t + f'(\hat{\epsilon}_t)}{\hat{\alpha}_t} - f'(\hat{\epsilon}_t)\right] \\
&- \bar{\nu}_t \hat{\alpha}_t (1-\hat{\alpha}_t) - \bar{\xi}_t \hat{\alpha}_t M_t = 0,
\end{aligned}$$

$$\begin{aligned}
-\dot{\bar{\nu}}_t + \bar{\nu}_t \hat{\epsilon}_t (1-2\hat{\alpha}_t) = M_t \lambda \hat{\epsilon}_t (R+E+I) - (1-\lambda)(1-M_t)\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t^2} \\
- \bar{\xi}_t \hat{\epsilon}_t M_t - r\bar{\nu}_t,
\end{aligned}$$

$$\begin{aligned}
-\dot{\bar{\xi}}_t + \bar{\xi}_t \hat{\alpha}_t \hat{\epsilon}_t = \left[\lambda \hat{\alpha}_t \hat{\epsilon}_t (R+E+I) + (1-\lambda) f'(\hat{\epsilon}_t)\hat{\epsilon}_t - f(\hat{\epsilon}_t) - \lambda \hat{\epsilon}_t c\right] \\
- (1-\lambda)\left[\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f(\hat{\epsilon}_t)\right] - r\bar{\xi}_t.
\end{aligned}$$

Differentiate the first condition with respect to time:

$$\begin{aligned}
&\dot{M}_t \left[\lambda \hat{\alpha}_t (R+E+I) + (1-\lambda) f''(\hat{\epsilon}_t)\hat{\epsilon}_t - \lambda f'(\hat{\epsilon}_t) - \lambda c\right] \\
&+ M_t \left[\lambda \dot{\hat{\alpha}}_t (R+E+I) + (1-\lambda) f'''(\hat{\epsilon}_t)\hat{\epsilon}_t \dot{\hat{\epsilon}}_t + (1-2\lambda) f''(\hat{\epsilon}_t)\dot{\hat{\epsilon}}_t\right] \\
&- \dot{M}_t (1-\lambda)\left[\frac{f''(\hat{\epsilon}_t)\hat{\epsilon}_t + f'(\hat{\epsilon}_t)}{\hat{\alpha}_t} - f'(\hat{\epsilon}_t)\right] \\
&+ (1-\lambda)(1-M_t)\left[\frac{f'''(\hat{\epsilon}_t)\hat{\epsilon}_t \dot{\hat{\epsilon}}_t + 2f''(\hat{\epsilon}_t)\dot{\hat{\epsilon}}_t}{\hat{\alpha}_t} - \frac{f''(\hat{\epsilon}_t)\hat{\epsilon}_t + f'(\hat{\epsilon}_t)}{\hat{\alpha}_t^2}\dot{\hat{\alpha}}_t - f''(\hat{\epsilon}_t)\dot{\hat{\epsilon}}_t\right] \\
&+ \hat{\alpha}_t (1-\hat{\alpha}_t)\left[-\dot{\bar{\nu}}_t + \hat{\epsilon}_t \bar{\nu}_t (1-2\hat{\alpha}_t)\right] - \dot{\bar{\xi}}_t \hat{\alpha}_t M_t - \bar{\xi}_t \dot{\hat{\alpha}}_t M_t - \bar{\xi}_t \hat{\alpha}_t \dot{M}_t = 0.
\end{aligned}$$

Combine it with the second necessary condition and simplify:

$$\begin{aligned}
&\dot{M}_t \left[\lambda \hat{\alpha}_t (R+E+I) + (1-\lambda) f''(\hat{\epsilon}_t)\hat{\epsilon}_t - \lambda f'(\hat{\epsilon}_t) - \lambda c\right] \\
&+ M_t \left[(1-\lambda) f'''(\hat{\epsilon}_t)\hat{\epsilon}_t \dot{\hat{\epsilon}}_t + (1-2\lambda) f''(\hat{\epsilon}_t)\dot{\hat{\epsilon}}_t\right] \\
&- \dot{M}_t (1-\lambda)\left[\frac{f''(\hat{\epsilon}_t)\hat{\epsilon}_t + f'(\hat{\epsilon}_t)}{\hat{\alpha}_t} - f'(\hat{\epsilon}_t)\right] \\
&+ (1-\lambda)(1-M_t)\left[\frac{f'''(\hat{\epsilon}_t)\hat{\epsilon}_t \dot{\hat{\epsilon}}_t + 2f''(\hat{\epsilon}_t)\dot{\hat{\epsilon}}_t}{\hat{\alpha}_t} - \frac{f''(\hat{\epsilon}_t)\hat{\epsilon}_t + f'(\hat{\epsilon}_t)}{\hat{\alpha}_t^2}\dot{\hat{\alpha}}_t - f''(\hat{\epsilon}_t)\dot{\hat{\epsilon}}_t\right] \\
&- \hat{\alpha}_t (1-\hat{\alpha}_t)(1-\lambda)(1-M_t)\frac{f'(\hat{\epsilon}_t)\hat{\epsilon}_t + rc}{\hat{\alpha}_t^2}
\end{aligned}$$

$$- \hat{\alpha}_t \left(1 - \hat{\alpha}_t\right) r \bar{\nu}_t + \hat{\alpha}_t M_t \left[-\dot{\bar{\xi}}_t + \bar{\xi}_t \hat{\epsilon}_t \hat{\alpha}_t \right] = 0.$$

Next, combine this result with the third necessary condition:

$$\begin{aligned}
&\hat{\alpha}_t M_t \left[\lambda f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - \lambda f\left(\hat{\epsilon}_t\right) + (1-\lambda) \frac{\left(1-\hat{\alpha}_t\right) f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t^2 - rc}{\hat{\alpha}_t} \right] \\
&+ M_t \left[(1-\lambda) f'''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + (1-2\lambda) f''\left(\hat{\epsilon}_t\right) \right] \dot{\hat{\epsilon}}_t \\
&+ (1-\lambda)\left(1-M_t\right) \left[\frac{f'''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + 2 f''\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t} - f''\left(\hat{\epsilon}_t\right) \right] \dot{\hat{\epsilon}}_t \\
&+ \left(1-\hat{\alpha}_t\right)(1-\lambda)\left(1-M_t\right) \frac{f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t^2 - rc}{\hat{\alpha}_t} \\
&- r \left[\hat{\alpha}_t \left(1-\hat{\alpha}_t\right) \bar{\nu}_t + \hat{\alpha}_t M_t \bar{\xi}_t \right] = 0.
\end{aligned}$$

Now, combine it with the first necessary condition and divide both sides by M_t:

$$\begin{aligned}
&\hat{\alpha}_t \left[\lambda f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - \lambda f\left(\hat{\epsilon}_t\right) + (1-\lambda) \frac{\left(1-\hat{\alpha}_t\right) f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t^2 - rc}{\hat{\alpha}_t} \right] \\
&+ \left[(1-\lambda) f'''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + (1-2\lambda) f''\left(\hat{\epsilon}_t\right) \right] \dot{\hat{\epsilon}}_t \\
&+ (1-\lambda) \frac{1-M_t}{M_t} \left[\frac{f'''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + 2 f''\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t} - f''\left(\hat{\epsilon}_t\right) \right] \dot{\hat{\epsilon}}_t \\
&+ \left(1-\hat{\alpha}_t\right)(1-\lambda) \frac{1-M_t}{M_t} \frac{f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t^2 - rc}{\hat{\alpha}_t} \\
&= r \left[\lambda \hat{\alpha}_t (R+E+I) + (1-\lambda) f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - \lambda f'\left(\hat{\epsilon}_t\right) - \lambda c \right] \\
&+ r(1-\lambda) \frac{1-M_t}{M_t} \left[\frac{f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t} - f'\left(\hat{\epsilon}_t\right) \right].
\end{aligned}$$

It is easy to establish that when $\alpha_0 < 1$,

$$M_t = \frac{1-\alpha_0}{1-\alpha_t},$$

therefore,

$$\frac{1-M_t}{M_t} = \frac{\alpha_0 - \alpha_t}{1-\alpha_t} \frac{1-\alpha_t}{1-\alpha_0} = \frac{\alpha_0 - \alpha_t}{1-\alpha_0}.$$

Thus

$$\hat{\alpha}_t \left[\lambda f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - \lambda f\left(\hat{\epsilon}_t\right) + (1-\lambda) \frac{\left(1-\hat{\alpha}_t\right) f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t^2 - rc}{\hat{\alpha}_t} \right]$$

$$
\begin{aligned}
&+\left[(1-\lambda) f'''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t+(1-2\lambda) f''\left(\hat{\epsilon}_t\right)\right] \dot{\hat{\epsilon}}_t \\
&+(1-\lambda) \frac{\alpha_0-\hat{\alpha}_t}{1-\alpha_0}\left[\frac{f'''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t+2 f''\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f''\left(\hat{\epsilon}_t\right)\right] \dot{\hat{\epsilon}}_t \\
&+\left(1-\hat{\alpha}_t\right)(1-\lambda) \frac{\alpha_0-\hat{\alpha}_t}{1-\alpha_0} \frac{f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t^2-r c}{\hat{\alpha}_t} \\
=\; & r\left[\lambda \hat{\alpha}_t(R+E+I)+(1-\lambda) f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t-\lambda f'\left(\hat{\epsilon}_t\right)-\lambda c\right] \\
&+r(1-\lambda) \frac{\alpha_0-\hat{\alpha}_t}{1-\alpha_0}\left[\frac{f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t+f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f'\left(\hat{\epsilon}_t\right)\right] .
\end{aligned}
$$

Notice that $\lambda = 1$ produces the first best solution. This is because I moved all the payments to $t = 0$, establishing

$$M_0 \mu + (1-\lambda) = 0,$$

where $M_0 = 1$. This way, when $\lambda = 1$, the budget constraint multiplier, $\mu = 0$, indicating that the entrepreneur does not need funds at all. In the actual equilibrium I, therefore, expect $\lambda > 1$.

To have the unique solution, I need to establish the boundary condition. Assign $\hat{\epsilon}_t = 0$ and $\dot{\epsilon}_t = 0$ and find $\underline{\alpha}$ that satisfied the equilibrium condition:

$$r\left[\lambda \underline{\alpha}(R+E+I)-\lambda c\right]+r c(1-\lambda)+r c(1-\underline{\alpha})(1-\lambda) \frac{\alpha_0-\underline{\alpha}}{\underline{\alpha}\left(1-\alpha_0\right)}=0,$$

or

$$\underline{\alpha}^2\left[(R+E+I) \lambda\left(1-\alpha_0\right)+c(1-\lambda)\right]-\underline{\alpha} c\left[2 \alpha_0+\lambda\left(1-3 \alpha_0\right)\right]+\alpha_0 c(1-\lambda)=0.$$

Thus the lower bound on belief level is

$$\underline{\alpha}=\frac{c\left[2 \alpha_0+\lambda\left(1-3 \alpha_0\right)\right]+\sqrt{1-\alpha_0} \cdot \sqrt{\lambda^2\left(1-\alpha_0\right) c^2-4 \alpha_0(1-\lambda)\left[\lambda \bar{R} c+(1-2 \lambda) c^2\right]}}{2 \bar{R} \lambda\left(1-\alpha_0\right)+2 c(1-\lambda)},$$

where $\bar{R} = R+E+I$ is the total social surplus produced at the moment the experiment succeeds. This is the lower bound on the entrepreneur's belief level.

II.C.5. Properties of the Best Contract

Important properties of the best equilibrium contract and the experimentation path induced by the best contract are:

Funds are provided unconditionally : this is the result of trying to solve the entrepreneur's problem derived in Appendix II.C.3 by directly controlling for the funding rate. It produces necessary condition

$$M_t \mu = \lambda - 1,$$

which must be satisfied for every t. This is impossible to do as M_t decreases over time given positive experimentation rates. However, this condition hints that by reallocating funds to the earlier periods the entrepreneur can benefit as it will allow her to relax one or both of the constraints and thus improve her payoffs. Therefore, the entrepreneur is better off if she moves all of the funds to $t = 0$, that is, if she requests an upfront payment of

$$P = \int_0^\infty e^{-rt} M_t \hat{e}_t \, \mathrm{d}t,$$

or equivalently, asks for any other unconditional funding scheme, which guarantees the present value of P. In terms of multipliers it means that

$$\mu = -\left(1 - \lambda\right),$$

because $M_0 = 1$. Unconditional payments mean that the entrepreneur receives funds no matter what, even if the project succeeds. This way, the money is treated as given, and it does not affect the entrepreneur's second-phase decisions to experiment, which become entirely driven by the shares.

Experimentation rates are inefficient : this follows directly from analyzing the differential equation that describes the equilibrium experimentation path. Rewritten in terms of the equilibrium policy function, $\hat{\epsilon}(\hat{\alpha})$, it is

$$\begin{aligned}
&\hat{\alpha}\left[\lambda f'(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})-\lambda f(\hat{\epsilon}(\hat{\alpha}))+(1-\lambda)\frac{(1-\hat{\alpha})f''(\hat{\epsilon}(\hat{\alpha}))(\hat{\epsilon}(\hat{\alpha}))^2-rc}{\hat{\alpha}}\right]\\
&+(1-\hat{\alpha})(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\frac{f''(\hat{\epsilon}(\hat{\alpha}))(\hat{\epsilon}(\hat{\alpha}))^2-rc}{\hat{\alpha}}\\
&-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})\left[(1-\lambda)f'''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+(1-2\lambda)f''(\hat{\epsilon}(\hat{\alpha}))\right]\hat{\epsilon}'(\hat{\alpha})\\
&-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\left[\frac{f'''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+2f''(\hat{\epsilon}(\hat{\alpha}))}{\hat{\alpha}}-f''(\hat{\epsilon}(\hat{\alpha}))\right]\hat{\epsilon}'(\hat{\alpha})\\
&=r\left[\lambda\hat{\alpha}\bar{R}+(1-\lambda)f''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})-\lambda f'(\hat{\epsilon}(\hat{\alpha}))-\lambda c\right]\\
&+r(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\left[\frac{f''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+f'(\hat{\epsilon}(\hat{\alpha}))}{\hat{\alpha}}-f'(\hat{\epsilon}(\hat{\alpha}))\right].
\end{aligned} \tag{II.e}$$

When

$$\mu=-(1-\lambda)=0,$$

which means that the budget constraint is no longer binding, (II.e) degenerates to

$$\hat{\alpha}\left[f'(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})-f(\hat{\epsilon}(\hat{\alpha}))\right]+\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})f''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}'(\hat{\alpha})=r\left[\hat{\alpha}\bar{R}-f'(\hat{\epsilon}(\hat{\alpha}))-c\right],$$

which is exactly the first best experimentation path. The boundary condition,

$$\hat{\epsilon}\left(\frac{c[2\alpha_0+\lambda(1-3\alpha_0)]+\sqrt{(1-\alpha_0)\left[\lambda^2(1-\alpha_0)c^2-4\alpha_0(1-\lambda)\left(\lambda\bar{R}c+(1-2\lambda)c^2\right)\right]}}{2\bar{R}\lambda(1-\alpha_0)+2c(1-\lambda)}\right)=0, \tag{II.f}$$

also degenerates to

$$\hat{\epsilon}\left(\frac{c}{\bar{R}}\right)=0.$$

It means that the only way to have the efficient experimentation path is to have no budget constraint. However, the entrepreneur needs funds. That is why she asks the investor to participate. Therefore, the best experimentation path is inefficient.

Externalities that the project produces for the parties will be internalized : there are two parameters that describe the externalities in this model, E, and I. The entrepreneur's externality, E is received by the entrepreneur in full conditional on the project success. Similarly, the investor receives I when the project succeeds. Despite that these parts of the surplus are not sharable per se, given transferable utilities it is possible to write a contract in such a way that these externalities will be internalized. In fact, this is exactly what happens in the best equilibrium as the funding path, (II.e), and the boundary condition, (II.f), do not depend on E and I individually and only depend on the total surplus,

$$\bar{R} = R + E + I.$$

As long as the total surplus is the same, the distribution of the externalities and sharable part R does not matter.

Some projects are not worth the risk : if $\alpha_0 \leqslant \frac{2c}{\bar{R}}$, then the project will not be undertaken. Suppose that there is no opportunity to work on the project in the future, so that current incentives to experiment are the highest as there will be no second change to succeed. For the project to be worked on at phase two, it must be the case that

$$\hat{\alpha}_0 \hat{\epsilon}_0 \left(s_0 R + E\right) - f\left(\hat{\epsilon}_0\right) - \hat{\epsilon}_0 c \geqslant 0.$$

It will have to be the case that at least

$$s_0 R \geqslant \frac{c}{\hat{\alpha}_0} - E.$$

For the investor to fund such a project it must be the case that

$$\hat{\alpha}_0 \hat{\epsilon}_0 \left[\left(1 - s_0\right) R + I\right) - \hat{\epsilon}_0 c \geqslant 0,$$

or

$$s_0 R \leqslant R + I - \frac{c}{\hat{\alpha}_0}.$$

Thus for the opportunity for such a share to exist, it must be the case that

$$R + I - \frac{c}{\hat{\alpha}_0} \geqslant \frac{c}{\hat{\alpha}_0} - E,$$

or

$$\hat{\alpha}_0 \geqslant \frac{2c}{R + I + E}.$$

Otherwise, the parties will not be able to agree even on a single experiment, let alone the whole continuum of future experimentation sessions.

Experimenting never stops : experiments only stop if success happens, otherwise, experimentation rate is always positive.

Take the policy function, (II.e), and write is as

$$\begin{aligned}
&\hat{\epsilon}'(\hat{\alpha})\Bigg[-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})\left[(1-\lambda)f'''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+(1-2\lambda)f''(\hat{\epsilon}(\hat{\alpha}))\right] \\
&-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\left[\frac{f'''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+2f''(\hat{\epsilon}(\hat{\alpha}))}{\hat{\alpha}}-f''(\hat{\epsilon}(\hat{\alpha}))\right]\Bigg] \\
&=r\left[\lambda\hat{\alpha}\bar{R}+(1-\lambda)f''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})-\lambda f'(\hat{\epsilon}(\hat{\alpha}))-\lambda c\right] \\
&+r(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\left[\frac{f''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+f'(\hat{\epsilon}(\hat{\alpha}))}{\hat{\alpha}}-f'(\hat{\epsilon}(\hat{\alpha}))\right] \\
&-\hat{\alpha}\left[\lambda f'(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})-\lambda f(\hat{\epsilon}(\hat{\alpha}))+(1-\lambda)\frac{(1-\hat{\alpha})f''(\hat{\epsilon}(\hat{\alpha}))(\hat{\epsilon}(\hat{\alpha}))^2-rc}{\hat{\alpha}}\right] \\
&-(1-\hat{\alpha})(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\frac{f''(\hat{\epsilon}(\hat{\alpha}))(\hat{\epsilon}(\hat{\alpha}))^2-rc}{\hat{\alpha}}.
\end{aligned}$$

Then

$$\hat{\epsilon}'(\hat{\alpha})=\frac{X(\hat{\alpha})}{Y(\hat{\alpha})},$$

where

$$\begin{aligned}
X(\hat{\alpha})\equiv\ &r\left[\lambda\hat{\alpha}\bar{R}+(1-\lambda)f''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})-\lambda f'(\hat{\epsilon}(\hat{\alpha}))-\lambda c\right] \\
&+r(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\left[\frac{f''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+f'(\hat{\epsilon}(\hat{\alpha}))}{\hat{\alpha}}-f'(\hat{\epsilon}(\hat{\alpha}))\right] \\
&-\hat{\alpha}\left[\lambda f'(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})-\lambda f(\hat{\epsilon}(\hat{\alpha}))+(1-\lambda)\frac{(1-\hat{\alpha})f''(\hat{\epsilon}(\hat{\alpha}))(\hat{\epsilon}(\hat{\alpha}))^2-rc}{\hat{\alpha}}\right] \\
&-(1-\hat{\alpha})(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\frac{f''(\hat{\epsilon}(\hat{\alpha}))(\hat{\epsilon}(\hat{\alpha}))^2-rc}{\hat{\alpha}},
\end{aligned}$$

and

$$\begin{aligned}
Y(\hat{\alpha})\equiv\ &-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})\left[(1-\lambda)f'''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+(1-2\lambda)f''(\hat{\epsilon}(\hat{\alpha}))\right] \\
&-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})(1-\lambda)\frac{\alpha_0-\hat{\alpha}}{1-\alpha_0}\left[\frac{f'''(\hat{\epsilon}(\hat{\alpha}))\hat{\epsilon}(\hat{\alpha})+2f''(\hat{\epsilon}(\hat{\alpha}))}{\hat{\alpha}}-f''(\hat{\epsilon}(\hat{\alpha}))\right].
\end{aligned}$$

It was determined in Appendix C that, for a normally behaving policy function, it is enough to establish that the policy function has a finite derivative at the boundary level of belief, $\underline{\alpha}$, for the experiments to only stop if success happens and otherwise continue forever. Hence, my goal is to show that

$$\lim_{\hat{\alpha}\to\underline{\alpha}} \hat{\epsilon}'\left(\hat{\alpha}\right) < \infty.$$

If I plug $\underline{\alpha}$ directly into the expression,

$$\hat{\epsilon}'\left(\underline{\alpha}\right) = \frac{X\left(\underline{\alpha}\right)}{Y\left(\underline{\alpha}\right)} = \frac{0}{0},$$

I will get the indeterminacy because $\hat{\epsilon}\left(\underline{\alpha}\right) = 0$. Hence, I use the L'Hôpital's rule and differentiate the numerator and the denominator with respect to $\hat{\alpha}$ and then assign $\hat{\alpha} = \underline{\alpha}$ to the results. The derivative of the numerator valued at $\underline{\alpha}$ is

$$X'\left(\underline{\alpha}\right) = r\left[\lambda\bar{R} - (2\lambda - 1) f''(0)\,\hat{\epsilon}'\left(\underline{\alpha}\right)\right] - r(\lambda - 1)\frac{\alpha_0 - \underline{\alpha}}{1 - \alpha_0}\frac{2 - \underline{\alpha}}{\underline{\alpha}} f''(0)\,\hat{\epsilon}'\left(\underline{\alpha}\right)$$
$$+ rc\frac{\lambda - 1}{1 - \alpha_0}\frac{\alpha_0 - \underline{\alpha}^2}{\underline{\alpha}^2},$$

The derivative of the denominator valued at $\underline{\alpha}$ is

$$Y'\left(\underline{\alpha}\right) = \underline{\alpha}\left(1 - \underline{\alpha}\right) f''(0)\,\hat{\epsilon}'\left(\underline{\alpha}\right)\left[\lambda + (\lambda - 1)\frac{\left(2\alpha_0 - \underline{\alpha}\right)\left(1 - \underline{\alpha}\right)}{\underline{\alpha}\left(1 - \alpha_0\right)}\right].$$

Therefore,

$$\hat{\epsilon}'\left(\underline{\alpha}\right) = \frac{X'\left(\underline{\alpha}\right)}{X'\left(\underline{\alpha}\right)} = \frac{r\left[\frac{\lambda\bar{R}}{f''(0)\hat{\epsilon}'(\underline{\alpha})} - (2\lambda - 1)\right] - r(\lambda - 1)\frac{\alpha_0 - \underline{\alpha}}{1 - \alpha_0}\frac{2 - \underline{\alpha}}{\underline{\alpha}} + \frac{rc}{f''(0)\hat{\epsilon}'(\underline{\alpha})}\frac{\lambda - 1}{1 - \alpha_0}\frac{\alpha_0 - \underline{\alpha}^2}{\underline{\alpha}^2}}{\underline{\alpha}\left(1 - \underline{\alpha}\right)\left[\lambda + (\lambda - 1)\frac{(2\alpha_0 - \underline{\alpha})(1 - \underline{\alpha})}{\underline{\alpha}(1 - \alpha_0)}\right]}.$$

Having $\hat{\epsilon}'\left(\underline{\alpha}\right) = \infty$ is inconsistent with this expression. So it must be the case that the policy function has a finite derivative at the boundary belief level. Thus the experiments continue until success happens of forever.

II.C.6. The Contract with Conditional Funding

The maximization problem the entrepreneur is facing at phase one of the game when the project is financed conditionally on no success over time and the investor provides exactly the amount of funds the entrepreneur expects to spend on a particular moment in time is

$$J = \max_{(\hat{\epsilon}_t, t\geqslant 0),\lambda} \int_0^\infty e^{-tr}\Big[M_t\left(\lambda\hat{\alpha}_t\hat{\epsilon}_t\left(R + E + I\right) + (1 - \lambda) f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right) - \lambda\hat{\epsilon}_t c\right)$$

$$+\left(1-\lambda\right)\left(1-M_t\right)\left(\frac{f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+rc}{\hat{\alpha}_t}-f\left(\hat{\epsilon}_t\right)+\hat{\epsilon}_t c\right)\Bigg]\,\mathrm{d}t$$

$$\text{subject to: } \dot{\hat{\alpha}}_t=-\hat{\alpha}_t\hat{\epsilon}_t\left(1-\hat{\alpha}_t\right),$$

$$\dot{M}_t=-\hat{\alpha}_t\hat{\epsilon}_t M_t,$$

$$\alpha_0 \text{ given, } \alpha_0\in\left[0,1\right],$$

$$M_0=1.$$

This is a dynamic control problem. I will solve it using the Pontryagin's Maximum Principle.

Assign multipliers ν_t and ξ_t to the first and the second constraint from this problem, respectively. Assume that the end values for the state variables are known, and write the Lagrangian:

$$\begin{aligned}\mathcal{L}=\int_0^{\infty}\Bigg[&e^{-tr}M_t\left[\lambda\hat{\alpha}_t\hat{\epsilon}_t\left(R+E+I\right)+\left(1-\lambda\right)f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-f\left(\hat{\epsilon}_t\right)-\lambda\hat{\epsilon}_t c\right]\\&+e^{-tr}\left(1-\lambda\right)\left(1-M_t\right)\left[\frac{f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+rc}{\hat{\alpha}_t}-f\left(\hat{\epsilon}_t\right)+\hat{\epsilon}_t c\right]\\&-\nu_t\left[\hat{\alpha}_t\hat{\epsilon}_t\left(1-\hat{\alpha}_t\right)+\dot{\hat{\alpha}}_t\right]-\xi_t\left[\hat{\alpha}_t\hat{\epsilon}_t M_t+\dot{M}_t\right]\Bigg]\,\mathrm{d}t.\end{aligned}$$

It looks remarkably similar to the Lagrangian produced for the problem of finding the induced experimentation path under the terms of the best contract. The only difference is the inclusion of the $\hat{\epsilon}_t c$ term on the second line. I can jump straight to writing the Hamiltonian:

$$\begin{aligned}\mathcal{H}_t=\,&e^{-tr}M_t\left[\lambda\hat{\alpha}_t\hat{\epsilon}_t\left(R+E+I\right)+\left(1-\lambda\right)f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-f\left(\hat{\epsilon}_t\right)-\lambda\hat{\epsilon}_t c\right]\\&+e^{-tr}\left(1-\lambda\right)\left(1-M_t\right)\left[\frac{f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+rc}{\hat{\alpha}_t}-f\left(\hat{\epsilon}_t\right)+\hat{\epsilon}_t c\right]\\&-\nu_t\hat{\alpha}_t\hat{\epsilon}_t\left(1-\hat{\alpha}_t\right)-\xi_t\hat{\alpha}_t\hat{\epsilon}_t M_t.\end{aligned}$$

From this point, it is clear what the the necessary conditions are:

$$\begin{aligned}\frac{\partial\mathcal{H}_t}{\partial\hat{\epsilon}_t}=\,&e^{-tr}M_t\left[\lambda\hat{\alpha}_t\left(R+E+I\right)+\left(1-\lambda\right)f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-\lambda f'\left(\hat{\epsilon}_t\right)-\lambda c\right]\\&+e^{-tr}\left(1-\lambda\right)\left(1-M_t\right)\left[\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f'\left(\hat{\epsilon}_t\right)+c\right]\\&-\nu_t\hat{\alpha}_t\left(1-\hat{\alpha}_t\right)-\xi_t\hat{\alpha}_t M_t=0,\end{aligned}$$

$$\dot{\nu}_t=-\frac{\partial\mathcal{H}_t}{\partial\hat{\alpha}_t}=-e^{-tr}M_t\lambda\hat{\epsilon}_t\left(R+E+I\right)+e^{-tr}\left(1-\lambda\right)\left(1-M_t\right)\frac{f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+rc}{\hat{\alpha}_t^2}$$

$$+ \nu_t \hat{\epsilon}_t \left(1 - 2\hat{\alpha}_t\right) + \xi_t \hat{\epsilon}_t M_t,$$

$$\begin{aligned}\dot{\xi}_t = -\frac{\partial \mathcal{H}_t}{\partial M_t} = -e^{-tr}\left[\lambda \hat{\alpha}_t \hat{\epsilon}_t \left(R + E + I\right) + \left(1 - \lambda\right) f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right) - \lambda \hat{\epsilon}_t c\right] \\ + e^{-tr}\left(1 - \lambda\right)\left[\frac{f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f\left(\hat{\epsilon}_t\right) + \hat{\epsilon}_t c\right] \\ + \xi_t \hat{\alpha}_t \hat{\epsilon}_t,\end{aligned}$$

as well as laws of motion and the boundary conditions.

Define

$$\begin{aligned}\nu_t &\equiv e^{-rt}\bar{\nu}_t, &\Rightarrow&& \dot{\nu}_t &= -re^{-rt}\bar{\nu}_t + e^{-rt}\dot{\bar{\nu}}_t, \\ \xi_t &\equiv e^{-rt}\bar{\xi}_t, &\Rightarrow&& \dot{\xi}_t &= -re^{-rt}\bar{\xi}_t + e^{-rt}\dot{\bar{\xi}}_t.\end{aligned}$$

Using these new costate variables, rewrite the necessary conditions:

$$\begin{aligned}M_t\left[\lambda \hat{\alpha}_t \left(R + E + I\right) + \left(1 - \lambda\right) f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - \lambda f'\left(\hat{\epsilon}_t\right) - \lambda c\right] \\ + \left(1 - \lambda\right)\left(1 - M_t\right)\left[\frac{f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t} - f'\left(\hat{\epsilon}_t\right) + c\right] \\ - \bar{\nu}_t \hat{\alpha}_t \left(1 - \hat{\alpha}_t\right) - \bar{\xi}_t \hat{\alpha}_t M_t = 0,\end{aligned}$$

$$\begin{aligned}-\dot{\bar{\nu}}_t + \bar{\nu}_t \hat{\epsilon}_t \left(1 - 2\hat{\alpha}_t\right) = M_t \lambda \hat{\epsilon}_t \left(R + E + I\right) - \left(1 - \lambda\right)\left(1 - M_t\right) \frac{f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + rc}{\hat{\alpha}_t^2} \\ - \bar{\xi}_t \hat{\epsilon}_t M_t - r\bar{\nu}_t,\end{aligned}$$

$$\begin{aligned}-\dot{\bar{\xi}}_t + \bar{\xi}_t \hat{\alpha}_t \hat{\epsilon}_t = \left[\lambda \hat{\alpha}_t \hat{\epsilon}_t \left(R + E + I\right) + \left(1 - \lambda\right) f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - f\left(\hat{\epsilon}_t\right) - \lambda \hat{\epsilon}_t c\right] \\ - \left(1 - \lambda\right)\left[\frac{f'\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + rc}{\hat{\alpha}_t} - f\left(\hat{\epsilon}_t\right) + \hat{\epsilon}_t c\right] - r\bar{\xi}_t.\end{aligned}$$

Differentiate the first condition with respect to time:

$$\begin{aligned}\dot{M}_t\left[\lambda \hat{\alpha}_t \left(R + E + I\right) + \left(1 - \lambda\right) f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t - \lambda f'\left(\hat{\epsilon}_t\right) - \lambda c\right] \\ + M_t\left[\lambda \dot{\hat{\alpha}}_t \left(R + E + I\right) + \left(1 - \lambda\right) f'''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t \dot{\hat{\epsilon}}_t + \left(1 - 2\lambda\right) f''\left(\hat{\epsilon}_t\right) \dot{\hat{\epsilon}}_t\right] \\ - \dot{M}_t\left(1 - \lambda\right)\left[\frac{f''\left(\hat{\epsilon}_t\right) \hat{\epsilon}_t + f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t} - f'\left(\hat{\epsilon}_t\right) + c\right]\end{aligned}$$

$$+\left(1-\lambda\right)\left(1-M_t\right)\left[\frac{f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t\dot{\hat{\epsilon}}_t+2f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t}{\hat{\alpha}_t}-\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t^2}\dot{\hat{\alpha}}_t-f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t\right]$$
$$+\hat{\alpha}_t\left(1-\hat{\alpha}_t\right)\left[-\dot{\bar{\nu}}_t+\hat{\epsilon}_t\bar{\nu}_t\left(1-2\hat{\alpha}_t\right)\right]-\dot{\bar{\xi}}_t\hat{\alpha}_tM_t-\bar{\xi}_t\dot{\hat{\alpha}}_tM_t-\bar{\xi}_t\hat{\alpha}_t\dot{M}_t=0.$$

Combine it with the second necessary condition and simplify:

$$\begin{aligned}
&\dot{M}_t\left[\lambda\hat{\alpha}_t\left(R+E+I\right)+\left(1-\lambda\right)f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-\lambda f'\left(\hat{\epsilon}_t\right)-\lambda c\right]\\
&+M_t\left[\left(1-\lambda\right)f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t\dot{\hat{\epsilon}}_t+\left(1-2\lambda\right)f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t\right]\\
&-\dot{M}_t\left(1-\lambda\right)\left[\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f'\left(\hat{\epsilon}_t\right)+c\right]\\
&+\left(1-\lambda\right)\left(1-M_t\right)\left[\frac{f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t\dot{\hat{\epsilon}}_t+2f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t}{\hat{\alpha}_t}-\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t^2}\dot{\hat{\alpha}}_t-f''\left(\hat{\epsilon}_t\right)\dot{\hat{\epsilon}}_t\right]\\
&-\hat{\alpha}_t\left(1-\hat{\alpha}_t\right)\left(1-\lambda\right)\left(1-M_t\right)\frac{f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+rc}{\hat{\alpha}_t^2}\\
&-\hat{\alpha}_t\left(1-\hat{\alpha}_t\right)r\bar{\nu}_t+\hat{\alpha}_tM_t\left[-\dot{\bar{\xi}}_t+\bar{\xi}_t\hat{\epsilon}_t\hat{\alpha}_t\right]=0.
\end{aligned}$$

Next, combine this result with the third necessary condition:

$$\begin{aligned}
&\hat{\alpha}_tM_t\left[\lambda f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-\lambda f\left(\hat{\epsilon}_t\right)+\left(1-\lambda\right)\frac{\left(1-\hat{\alpha}_t\right)f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t^2-rc}{\hat{\alpha}_t}\right]\\
&+M_t\left[\left(1-\lambda\right)f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+\left(1-2\lambda\right)f''\left(\hat{\epsilon}_t\right)\right]\dot{\hat{\epsilon}}_t\\
&+\left(1-\lambda\right)\left(1-M_t\right)\left[\frac{f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+2f''\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f''\left(\hat{\epsilon}_t\right)\right]\dot{\hat{\epsilon}}_t\\
&+\left(1-\hat{\alpha}_t\right)\left(1-\lambda\right)\left(1-M_t\right)\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t^2-rc}{\hat{\alpha}_t}\\
&-r\left[\hat{\alpha}_t\left(1-\hat{\alpha}_t\right)\bar{\nu}_t+\hat{\alpha}_tM_t\bar{\xi}_t\right]=0.
\end{aligned}$$

Now, combine it with the first necessary condition and divide both sides by M_t:

$$\begin{aligned}
&\hat{\alpha}_t\left[\lambda f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-\lambda f\left(\hat{\epsilon}_t\right)+\left(1-\lambda\right)\frac{\left(1-\hat{\alpha}_t\right)f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t^2-rc}{\hat{\alpha}_t}\right]\\
&+\left[\left(1-\lambda\right)f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+\left(1-2\lambda\right)f''\left(\hat{\epsilon}_t\right)\right]\dot{\hat{\epsilon}}_t\\
&+\left(1-\lambda\right)\frac{1-M_t}{M_t}\left[\frac{f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+2f''\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f''\left(\hat{\epsilon}_t\right)\right]\dot{\hat{\epsilon}}_t\\
&+\left(1-\hat{\alpha}_t\right)\left(1-\lambda\right)\frac{1-M_t}{M_t}\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t^2-rc}{\hat{\alpha}_t}
\end{aligned}$$

$$= r\left[\lambda\hat{\alpha}_t\left(R+E+I\right)+\left(1-\lambda\right)f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-\lambda f'\left(\hat{\epsilon}_t\right)-\lambda c\right]$$
$$+ r\left(1-\lambda\right)\frac{1-M_t}{M_t}\left[\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f'\left(\hat{\epsilon}_t\right)+c\right].$$

Given that when $\alpha_0 < 1$,

$$M_t = \frac{1-\alpha_0}{1-\alpha_t},$$

and

$$\frac{1-M_t}{M_t} = \frac{\alpha_0-\alpha_t}{1-\alpha_0},$$

I can write

$$\hat{\alpha}_t\left[\lambda f'\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-\lambda f\left(\hat{\epsilon}_t\right)+\left(1-\lambda\right)\frac{\left(1-\hat{\alpha}_t\right)f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t^2-rc}{\hat{\alpha}_t}\right]$$
$$+\left[\left(1-\lambda\right)f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+\left(1-2\lambda\right)f''\left(\hat{\epsilon}_t\right)\right]\dot{\hat{\epsilon}}_t$$
$$+\left(1-\lambda\right)\frac{\alpha_0-\hat{\alpha}_t}{1-\alpha_0}\left[\frac{f'''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+2f''\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f''\left(\hat{\epsilon}_t\right)\right]\dot{\hat{\epsilon}}_t$$
$$+\left(1-\hat{\alpha}_t\right)\left(1-\lambda\right)\frac{\alpha_0-\hat{\alpha}_t}{1-\alpha_0}\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t^2-rc}{\hat{\alpha}_t}$$
$$= r\left[\lambda\hat{\alpha}_t\left(R+E+I\right)+\left(1-\lambda\right)f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t-\lambda f'\left(\hat{\epsilon}_t\right)-\lambda c\right]$$
$$+ r\left(1-\lambda\right)\frac{\alpha_0-\hat{\alpha}_t}{1-\alpha_0}\left[\frac{f''\left(\hat{\epsilon}_t\right)\hat{\epsilon}_t+f'\left(\hat{\epsilon}_t\right)}{\hat{\alpha}_t}-f'\left(\hat{\epsilon}_t\right)+c\right].$$

Again, $\lambda = 1$ produces the first best solution.

To establish the boundary condition and have a unique solution, I assign $\hat{\epsilon}_t = 0$ and $\dot{\epsilon}_t = 0$ and find $\underline{\alpha}$ that satisfied the equilibrium condition:

$$r\left[\lambda\underline{\alpha}\left(R+E+I\right)-\lambda c\right]+rc\left(1-\lambda\right)+rc\left(1-\lambda\right)\frac{\alpha_0-\underline{\alpha}}{\underline{\alpha}\left(1-\alpha_0\right)}=0,$$

or

$$\underline{\alpha}^2\lambda\bar{R}\left(1-\alpha_0\right)-\underline{\alpha}c\left(\lambda+\alpha_0\left(1-2\lambda\right)\right)+c\left(1-\lambda\right)\alpha_0=0.$$

Thus the lower bound on belief level is

$$\underline{\alpha}=\frac{c\left[\lambda+\alpha_0\left(1-2\lambda\right)\right]+\sqrt{c^2\left(\lambda+\alpha_0\left(1-2\lambda\right)\right)^2-4\lambda\bar{R}\left(1-\alpha_0\right)c\left(1-\lambda\right)\alpha_0}}{2\lambda\bar{R}\left(1-\alpha_0\right)}.$$

This is the lower bound on the entrepreneur's belief level.

Chapter III

Appendices

III.A. From Discrete to Continuous Time

I begin with some important observations regarding the effort level. Since, in discrete time, I interpret $\epsilon_t d$ as the probability that the experiment conducted at time t succeeds and since I don't want the probability of one or higher to be feasible, the property that I need to satisfy for all t is

$$\epsilon_t d < 1.$$

Consequently, for any $d > 0$,

$$\lim_{\epsilon_t \to \frac{1}{d}} f_d\left(\epsilon_t\right) d = \infty.$$

Therefore, when $d \to 0$, $f_d \to f$, and so

$$\lim_{\epsilon_t \to \infty} f\left(\epsilon_t\right) = \infty,$$

which means that as the effort cost function in continuous time, I can use any function that is strictly increasing on $\mathbb{R}^+$, strictly convex, with all the other required properties inherited from function f_d.

Now, suppose that all the maximization problems are solved and we are describing the value functions in terms of the equilibrium values of the state and control variables:

$$V\left(\alpha_t, \hat{\alpha}_t\right) = \alpha_t \epsilon_t d s_t R - f_d\left(\epsilon_t\right) d + \gamma_t dc - \epsilon_t dc + \left(1 - \alpha_t \epsilon_t d\right) \delta V\left(\alpha_{t+d}, \hat{\alpha}_{t+d}\right),$$

$$W\left(\hat{\alpha}_t\right) = \hat{\alpha}_t \hat{\epsilon}_t d\left(1 - s_t\right) R - \gamma_t dc + \left(1 - \hat{\alpha}_t \hat{\epsilon}_t d\right) \delta W\left(\hat{\alpha}_{t+d}\right).$$

These functions can be represented in the simplified form:

$$V_t = v_t d + (1 - \alpha_t \epsilon_t d)\, \delta V_{t+d},$$

$$W_t = w_t d + (1 - \hat{\alpha}_t \hat{\epsilon}_t d)\, \delta W_{t+d},$$

where

$$V_t \equiv V(\alpha_t, \hat{\alpha}_t), \qquad v_t \equiv \alpha_t \epsilon_t s_t R - f_d(\epsilon_t) + \gamma_t c - \epsilon_t c,$$

$$W_t \equiv W(\hat{\alpha}_t), \qquad w_t \equiv \hat{\alpha}_t \hat{\epsilon}_t (1 - s_t) R - \gamma_t c.$$

It is important to emphasize that given the previous discussion about function f_d and effort rate ϵ_t,

$$\lim_{d \to 0} \left[\alpha_t \epsilon_t s_t R - f_d(\epsilon_t) + \gamma_t c - \epsilon_t c\right] = \alpha_t \epsilon_t s_t R - f(\epsilon_t) + \gamma_t c - \epsilon_t c.$$

Since the two value functions now look similar enough, I will work with the first one and then will deal with the second one analogously.

Begin by expanding the value function:

$$V_t = v_t d + \delta (1 - \alpha_t \epsilon_t d)\, v_{t+d} d + \delta^2 (1 - \alpha_t \epsilon_t d)(1 - \alpha_{t+d} \epsilon_{t+d} d)\, v_{t+2d} d + \ldots$$

Since

$$\alpha_{t+d} = \frac{\alpha_t (1 - \epsilon_t d)}{1 - \alpha_t \epsilon_t d},$$

it is clear that

$$1 - \alpha_t \epsilon_t d = \frac{\alpha_t (1 - \epsilon_t d)(1 - \alpha_t \epsilon_t d)}{\alpha_t (1 - \epsilon_t d)} = \frac{\alpha_t}{\alpha_{t+d}} (1 - \epsilon_t d),$$

and that

$$(1 - \alpha_t \epsilon_t d)(1 - \alpha_{t+d} \epsilon_{t+d} d) = \frac{\alpha_t}{\alpha_{t+d}} (1 - \epsilon_t d) \frac{\alpha_{t+d}}{\alpha_{t+2d}} (1 - \epsilon_{t+d} d) = \frac{\alpha_t}{\alpha_{t+2d}} (1 - \epsilon_t d)(1 - \epsilon_{t+d} d).$$

By extrapolation,

$$\prod_{j=0}^{i-1} (1 - \alpha_{t+jd} \epsilon_{t+jd} d) = \frac{\alpha_t}{\alpha_{t+id}} \prod_{j=0}^{i-1} (1 - \epsilon_{t+jd} d),$$

and so

$$V_t = v_t d + d\sum_{i=1}^{\infty}\delta^i \frac{\alpha_t}{\alpha_{t+id}}\prod_{j=0}^{i-1}\left(1-\epsilon_{t+jd}d\right)v_{t+id}.$$

Since time advances in increments of d each period, after i periods, time will be $\tau = t + id$. Therefore,

$$i = \frac{\tau - t}{d}.$$

Define

$$\frac{1}{1+rd} \equiv \delta.$$

The limit of the value function as d goes to zero, is

$$\lim_{d\to 0} V_t = \int_t^{\infty}\lim_{d\to 0}\frac{\prod_{j=0}^{\frac{\tau-t}{d}-1}\left(1-\epsilon_{t+jd}d\right)}{\left(1+rd\right)^{\frac{\tau-t}{d}}}\frac{\alpha_t}{\alpha_\tau}v_\tau\,\mathrm{d}\tau.$$

Observe that

$$\begin{aligned}
\lim_{d\to 0}\frac{\prod_{j=0}^{\frac{\tau-t}{d}-1}\left(1-\epsilon_{t+jd}d\right)}{\left(1+rd\right)^{\frac{\tau-t}{d}}}\frac{\alpha_t}{\alpha_\tau} &= e^{\lim_{d\to 0}\ln\left(\frac{\prod_{j=0}^{\frac{\tau-t}{d}-1}\left(1-\epsilon_{t+jd}d\right)}{(1+rd)^{\frac{\tau-t}{d}}}\frac{\alpha_t}{\alpha_\tau}\right)} \\
&= e^{\lim_{d\to 0}\left[\ln\left(\prod_{j=0}^{\frac{\tau-t}{d}-1}\left(1-\epsilon_{t+jd}d\right)\right)-\frac{\tau-t}{d}\ln(1+rd)+\ln\alpha_t-\ln\alpha_\tau\right]} \\
&= e^{\lim_{d\to 0}\left[\sum_{j=0}^{\frac{\tau-t}{d}-1}\ln\left(1-\epsilon_{t+jd}d\right)-(\tau-t)\frac{\ln(1+rd)}{d}+\ln\alpha_t-\ln\alpha_\tau\right]} \\
&= e^{\lim_{d\to 0} d\sum_{j=0}^{\frac{\tau-t}{d}-1}\frac{\ln\left(1-\epsilon_{t+jd}d\right)}{d}-\lim_{d\to 0}(\tau-t)\frac{\ln(1+rd)}{d}+\ln\alpha_t-\ln\alpha_\tau} \\
&= e^{\int_t^{\tau}\lim_{d\to 0}\frac{\ln\left(1-\epsilon_\theta d\right)}{d}\,\mathrm{d}\theta-\lim_{d\to 0}(\tau-t)\frac{\ln(1+rd)}{d}+\ln\alpha_t-\ln\alpha_\tau}.
\end{aligned}$$

Using the L'Hôpital's rule,

$$\lim_{d\to 0}\frac{\ln\left(1-\epsilon_\theta d\right)}{d} = \lim_{d\to 0}\frac{-\epsilon_\theta}{1-\epsilon_\theta d} = -\epsilon_\theta,$$

and

$$\lim_{d\to 0}\left(\tau-t\right)\frac{\ln\left(1+rd\right)}{d} = \left(\tau-t\right)\lim_{d\to 0}\frac{r}{1+rd} = \left(\tau-t\right)r.$$

It is important to describe the continuous dynamics of α_τ:

$$\lim_{d\to 0}\frac{\alpha_{\tau+d}-\alpha_\tau}{d} = \lim_{d\to 0}\frac{\frac{\alpha_\tau(1-\epsilon_\tau d)}{1-\alpha_\tau\epsilon_\tau d}-\alpha_\tau}{d}$$

$$
\begin{aligned}
&= \lim_{d\to 0} \frac{\frac{\alpha_\tau - \alpha_\tau\epsilon_\tau d}{1-\alpha_\tau\epsilon_\tau d} - \frac{\alpha_\tau - \alpha_\tau^2\epsilon_\tau d}{1-\alpha_\tau\epsilon_\tau d}}{d} \\
&= \lim_{d\to 0} \alpha_\tau\epsilon_\tau \frac{\alpha_\tau - 1}{1-\alpha_\tau\epsilon_\tau d} \\
&= -\alpha_\tau\epsilon_\tau \left(1-\alpha_\tau\right).
\end{aligned}
$$

Therefore,

$$
\dot{\alpha_\tau} = -\alpha_\tau\epsilon_\tau \left(1-\alpha_\tau\right). \tag{III.a}
$$

Since $\dot{\ln x} = \frac{\dot{x}}{x}$, it is easy to find that

$$
\dot{\ln \alpha_\tau} = \frac{\dot{\alpha_\tau}}{\alpha_\tau} = -\epsilon_\tau \left(1-\alpha_\tau\right).
$$

Now, $\ln \alpha_\tau$ can be calculated by using the integrals:

$$
\ln \alpha_\tau = \ln \alpha_t + \int_t^\tau \dot{\ln \alpha_\theta}\, \mathrm{d}\theta = \ln \alpha_t - \int_t^\tau \epsilon_\theta \left(1-\alpha_\theta\right) \mathrm{d}\theta.
$$

Using this and all the previous results, it is straightforward to conclude that

$$
\begin{aligned}
\lim_{d\to 0} \frac{\prod_{j=0}^{\frac{\tau-t}{d}-1} \left(1-\epsilon_{t+jd}d\right)}{\left(1+rd\right)^{\frac{\tau-t}{d}}} \frac{\alpha_t}{\alpha_\tau} &= e^{-\int_t^\tau \epsilon_\theta\, \mathrm{d}\theta - (\tau-t)r + \int_t^\tau \epsilon_\theta (1-\alpha_\theta)\, \mathrm{d}\theta} \\
&= e^{-(\tau-t)r - \int_t^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta},
\end{aligned}
$$

and so the entrepreneur's value function in continuous time can be written as

$$
V_t = \int_t^\infty e^{-(\tau-t)r - \int_t^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau. \tag{III.b}
$$

Focus on the time interval between t and $t+h$:

$$
\begin{aligned}
V_t &= \int_t^{t+h} e^{-(\tau-t)r - \int_t^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau + \int_{t+h}^\infty e^{-(\tau-t)r - \int_t^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau \\
&= \int_t^{t+h} e^{-(\tau-t)r - \int_t^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau + \int_{t+h}^\infty e^{-(\tau-(t+h))r - hr - \int_{t+h}^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta - \int_t^{t+h} \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau \\
&= \int_t^{t+h} e^{-(\tau-t)r - \int_t^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau + e^{-hr - \int_t^{t+h} \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \int_{t+h}^\infty e^{-(\tau-(t+h))r - \int_{t+h}^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau \\
&= \int_t^{t+h} e^{-(\tau-t)r - \int_t^\tau \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} \upsilon_\tau\, \mathrm{d}\tau + e^{-hr - \int_t^{t+h} \alpha_\theta\epsilon_\theta\, \mathrm{d}\theta} V_{t+h}.
\end{aligned}
$$

Rearrange and divide by h:

$$\frac{1}{h}\int_t^{t+h} e^{-(\tau-t)r-\int_t^{\tau}\alpha_\theta\epsilon_\theta\,\mathrm{d}\theta}v_\tau\,\mathrm{d}\tau+\frac{e^{-hr-\int_t^{t+h}\alpha_\theta\epsilon_\theta\,\mathrm{d}\theta}V_{t+h}-V_t}{h}=0.$$

Take limits as $h\to 0$. With the aid of the L'Hôpital's and the Leibniz rules, I produce

$$rV_t=v_t-\alpha_t\epsilon_tV_t+\dot{V}_t.$$

Now, it is possible to restore the entrepreneur's value function:

$$\begin{aligned}
rV(\alpha_t,\hat{\alpha}_t)&=\alpha_t\epsilon_ts_tR-f(\epsilon_t)+\gamma_tc-\epsilon_tc-\alpha_t\epsilon_tV(\alpha_t,\hat{\alpha}_t)+\frac{\mathrm{d}V(\alpha_t,\hat{\alpha}_t)}{\mathrm{d}t}\\
&=\alpha_t\epsilon_ts_tR-f(\epsilon_t)+\gamma_tc-\epsilon_tc-\alpha_t\epsilon_tV(\alpha_t,\hat{\alpha}_t)+V_1(\alpha_t,\hat{\alpha}_t)\dot{\alpha}_t+V_2(\alpha_t,\hat{\alpha}_t)\dot{\hat{\alpha}}_t\\
&=\alpha_t\epsilon_ts_tR-f(\epsilon_t)+\gamma_tc-\epsilon_tc-\alpha_t\epsilon_t\left(V(\alpha_t,\hat{\alpha}_t)+(1-\alpha_t)V_1(\alpha_t,\hat{\alpha}_t)\right)\\
&-\hat{\alpha}_t\hat{\epsilon}_t(1-\hat{\alpha}_t)V_2(\alpha_t,\hat{\alpha}_t).
\end{aligned}$$

Notice that I used (III.a) here for both α_t and $\hat{\alpha}_t$.

Finally, I can drop the subscripts and write a maximization problem in the form of Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
rV(\alpha,\hat{\alpha})=\max_{\epsilon}\,[&\alpha\epsilon sR-f(\epsilon)+\gamma c-\epsilon c-\alpha\epsilon\left(V(\alpha,\hat{\alpha})+(1-\alpha)V_1(\alpha,\hat{\alpha})\right)\\
&-\hat{\alpha}\hat{\epsilon}(1-\hat{\alpha})V_2(\alpha,\hat{\alpha})]
\end{aligned}$$

subject to the participation constraint.

Let us restore the investor's value function in a similar fashion:

$$\begin{aligned}
rW(\hat{\alpha}_t)&=\hat{\alpha}_t\hat{\epsilon}_t(1-s_t)R-\gamma_tc-\hat{\alpha}_t\hat{\epsilon}_tW(\hat{\alpha}_t)+\frac{\mathrm{d}W(\hat{\alpha}_t)}{\mathrm{d}t}\\
&=\hat{\alpha}_t\hat{\epsilon}_t(1-s_t)R-\gamma_tc-\hat{\alpha}_t\hat{\epsilon}_t\left(W(\hat{\alpha}_t)+(1-\hat{\alpha}_t)W'(\hat{\alpha}_t)\right).
\end{aligned}$$

In the form of the HJB equation, the investor problem becomes

$$rW(\hat{\alpha})=\max_{s,\gamma}\left[\hat{\alpha}\hat{\epsilon}(1-s)R-\gamma c-\hat{\alpha}\hat{\epsilon}\left(W(\hat{\alpha})+(1-\hat{\alpha})W'(\hat{\alpha})\right)\right]$$

subject to the participation and incentive constraints. The incentive constraint in the form of the HJB equation becomes

$$\hat{\epsilon}=\arg\max_{\epsilon}\left[\hat{\alpha}\epsilon sR-f(\epsilon)+\gamma c-\epsilon c-\hat{\alpha}\epsilon\left(V(\hat{\alpha},\hat{\alpha})+(1-\hat{\alpha})V_1(\hat{\alpha},\hat{\alpha})\right)\right.$$

$$- \hat{\alpha}\hat{\epsilon}\left(1-\hat{\alpha}\right)V_2\left(\hat{\alpha},\hat{\alpha}\right)\right].$$

Using (III.b), it is easy to produce

$$V\left(\alpha_t,\hat{\alpha}_t\right)$$
$$= \int_t^\infty e^{-(\tau-t)r-\int_t^\tau \alpha_\theta\epsilon(\alpha_\theta,\hat{\alpha}_\theta)\,\mathrm{d}\theta}\left[\alpha_\tau\epsilon\left(\alpha_\tau,\hat{\alpha}_\tau\right)s\left(\hat{\alpha}_\tau\right)R - f\left(\epsilon\left(\alpha_\tau,\hat{\alpha}_\tau\right)\right)+\gamma\left(\hat{\alpha}_\tau\right)c-\epsilon\left(\alpha_\tau,\hat{\alpha}_\tau\right)c\right]\,\mathrm{d}\tau$$

and

$$W\left(\hat{\alpha}_t\right) = \int_t^\infty e^{-(\tau-t)r-\int_t^\tau \hat{\alpha}_\theta\hat{\epsilon}(\hat{\alpha}_\theta)\,\mathrm{d}\theta}\left[\hat{\alpha}_\tau\hat{\epsilon}\left(\hat{\alpha}_\tau\right)\left(1-s\left(\hat{\alpha}_\tau\right)\right)R-\gamma\left(\hat{\alpha}_\tau\right)c\right]\,\mathrm{d}\tau,$$

where functions $\epsilon\left(\alpha_t,\hat{\alpha}_t\right)$, $s\left(\hat{\alpha}_t\right)$, and $\gamma\left(\hat{\alpha}_t\right)$ are the policy functions (they will be different for different information environments), and $\hat{\epsilon}\left(\hat{\alpha}_\tau\right)=\epsilon\left(\hat{\alpha}_t,\hat{\alpha}_t\right)$ given that the incentive constraint is satisfied.

III.B. Evolution of Beliefs and Critical Funding Rate

This appendix is devoted to the development of the critical funding rate that is used to determine if the funding ever stops. The idea behind this is that if the actual funding rate is everywhere below the critical funding rate then the funds are provided indefinitely conditional on no success.

First, I describe the evolution of α_t in continuous time. We know that

$$\alpha_{t+d} = \frac{\alpha_t\left(1-\epsilon_t d\right)}{1-\alpha_t\epsilon_t d},$$

and that

$$\begin{aligned}\alpha_{t+2d} &= \frac{\alpha_{t+d}\left(1-\epsilon_{t+d}d\right)}{1-\alpha_{t+d}\epsilon_{t+d}d}\\ &= \frac{\frac{\alpha_t(1-\epsilon_t d)}{1-\alpha_t\epsilon_t d}\left(1-\epsilon_{t+d}d\right)}{1-\frac{\alpha_t(1-\epsilon_t d)}{1-\alpha_t\epsilon_t d}\epsilon_{t+d}d}\\ &= \frac{\alpha_t\left(1-\epsilon_t d\right)\left(1-\epsilon_{t+d}d\right)}{1-\alpha_t+\alpha_t\left(1-\epsilon_t d\right)\left(1-\epsilon_{t+d}d\right)}.\end{aligned}$$

So

$$\alpha_{t+jd} = \frac{\alpha_t\prod_{i=0}^{j-1}\left(1-\epsilon_{t+id}d\right)}{1-\alpha_t+\alpha_t\prod_{i=0}^{j-1}\left(1-\epsilon_{t+id}d\right)}.$$

Call $\tau \equiv t + jd$, then $j = \frac{\tau - t}{d}$, and so

$$\alpha_\tau = \frac{\alpha_t \prod_{i=0}^{\frac{\tau-t-d}{d}} \left(1 - \epsilon_{t+id} d\right)}{1 - \alpha_t + \alpha_t \prod_{i=0}^{\frac{\tau-t-d}{d}} \left(1 - \epsilon_{t+id} d\right)}.$$

Take limits:

$$\begin{aligned}
\lim_{d\to\infty} \alpha_\tau &= e^{\lim_{d\to\infty} \ln\left(\frac{\alpha_t \prod_{i=0}^{\frac{\tau-t-d}{d}} (1-\epsilon_{t+id}d)}{1-\alpha_t+\alpha_t \prod_{i=0}^{\frac{\tau-t-d}{d}} (1-\epsilon_{t+id}d)}\right)} \\
&= e^{\lim_{d\to\infty} \left[\ln \alpha_t + \sum_{i=0}^{\frac{\tau-t-d}{d}} \ln(1-\epsilon_{t+id}d) - \ln\left(1-\alpha_t+\alpha_t \prod_{i=0}^{\frac{\tau-t-d}{d}} (1-\epsilon_{t+id}d)\right)\right]} \\
&= e^{\ln \alpha_t + \lim_{d\to\infty} d \sum_{i=0}^{\frac{\tau-t-d}{d}} \frac{\ln\left(1-\epsilon_{t+id}d\right)}{d} - \lim_{d\to\infty} \ln\left(1-\alpha_t+\alpha_t \prod_{i=0}^{\frac{\tau-t-d}{d}} (1-\epsilon_{t+id}d)\right)} \\
&= e^{\ln \alpha_t - \int_t^\tau \epsilon_\theta \, \mathrm{d}\theta - \ln\left(1-\alpha_t + e^{\ln \alpha_t - \int_t^\tau \epsilon_\theta \, \mathrm{d}\theta}\right)} \\
&= \frac{\alpha_t e^{-\int_t^\tau \epsilon_\theta \, \mathrm{d}\theta}}{1 - \alpha_t + \alpha_t e^{-\int_t^\tau \epsilon_\theta \, \mathrm{d}\theta}},
\end{aligned}$$

in particular,

$$\alpha_t = \frac{\alpha_0 e^{-\int_0^t \epsilon_\theta \, \mathrm{d}\theta}}{1 - \alpha_0 + \alpha_0 e^{-\int_0^t \epsilon_\theta \, \mathrm{d}\theta}}.$$

Second, I develop the critical funding rate. It is done in full in Appendix C.

III.C. The First Best Scenario

To begin solving the first best case, I first derive the combined, or social, value function. Given the transferability of utilities, a simple sum of the value functions produces the desired result. The social planner does not need to worry about the incentives and individual participation as if there is a positive expected surplus to gain, sharing it will not be an issue. I denote the combined value function as $\mathcal{V}(\alpha_t)$. It depends only on state variable α_t, as in the first best environment $\hat{\alpha}_t = \alpha_t$ for every t.

Since

$$\mathcal{V}(\alpha) = V(\alpha, \alpha) + W(\alpha),$$

I will use (III.1) and (III.2) to produce

$$r\mathcal{V}(\alpha) = rV(\alpha, \alpha) + rW(\alpha)$$

$$= \alpha\epsilon s R - f(\epsilon) + \gamma c - \epsilon c - \alpha\epsilon \left(V(\alpha,\alpha) + (1-\alpha) V_1(\alpha,\alpha)\right) - \alpha\epsilon (1-\alpha) V_2(\alpha,\alpha)$$

$$+ \alpha\epsilon (1-s) R - \gamma c - \alpha\epsilon \left(W(\alpha) + (1-\alpha) W'(\alpha)\right)$$

$$= \alpha\epsilon R - f(\epsilon) - \epsilon c - \alpha\epsilon \left[V(\alpha,\alpha) + W(\alpha) + (1-\alpha)\left(V_1(\alpha,\alpha) + V_2(\alpha,\alpha) + W'(\alpha)\right)\right]$$

$$= \alpha\epsilon R - f(\epsilon) - \epsilon c - \alpha\epsilon \left(\mathcal{V}(\alpha) + (1-\alpha)\mathcal{V}'(\alpha)\right),$$

and so as a HJB equation, the first-best problem becomes

$$r\mathcal{V}(\alpha) = \max_{\epsilon} \left[\alpha\epsilon R - f(\epsilon) - \epsilon c - \alpha\epsilon \left(\mathcal{V}(\alpha) + (1-\alpha)\mathcal{V}'(\alpha)\right)\right].$$

The first order condition is

$$\alpha R - f'(\epsilon^*(\alpha)) - c - \alpha \left[\mathcal{V}'(\alpha)(1-\alpha) + \mathcal{V}(\alpha)\right] = 0,$$

where function $\epsilon^*(\alpha)$ is the first-best policy function.

The second order condition

$$-f''(\epsilon^*(\alpha)) \leqslant 0$$

is satisfied for all $\epsilon > 0$.

Multiplying both sides of the first order condition by $\epsilon^*(\alpha)$, expressing

$$\alpha\epsilon^*(\alpha) R - \epsilon^*(\alpha) c - \alpha\epsilon^*(\alpha) \left[\mathcal{V}'(\alpha)(1-\alpha) + \mathcal{V}(\alpha)\right] = f'(\epsilon^*(\alpha))\,\epsilon^*(\alpha),$$

and combining it with the HJB equation yields

$$r\mathcal{V}(\alpha) = f'(\epsilon^*(\alpha))\,\epsilon^*(\alpha) - f(\epsilon^*(\alpha)).$$

Since it must be true for any α, differentiating the both sides of this expression with respect to α produces

$$\begin{aligned} r\mathcal{V}'(\alpha) &= f''(\epsilon^*(\alpha))\,\epsilon^*(\alpha)\,\epsilon^{*\prime}(\alpha) + f'(\epsilon^*(\alpha))\,\epsilon^{*\prime}(\alpha) - f'(\epsilon^*(\alpha))\,\epsilon^{*\prime}(\alpha) \\ &= f''(\epsilon^*(\alpha))\,\epsilon^*(\alpha)\,\epsilon^{*\prime}(\alpha). \end{aligned}$$

Notice that if $\mathcal{V}'(\alpha) > 0$, then $\epsilon^{*\prime}(\alpha) > 0$ for any positive $\epsilon^*(\alpha)$ because $f''(x) > 0$ if $x > 0$ (strict convexity).

To show that $\mathcal{V}'(\alpha_t) > 0$, it is useful to write the value function in the form similar to (III.b). In this sense, it is a sum of two value functions, (III.3) and (III.4), under the assumption that $\hat{\alpha}_t = \alpha_t$ for all t:

$$\mathcal{V}(\alpha_t) = \int_t^\infty e^{-(\tau-t)r-\int_t^\tau \alpha_\theta \epsilon^*(\alpha_\theta)\,\mathrm{d}\theta} \left[\alpha_\tau \epsilon^*(\alpha_\tau) R - f(\epsilon^*(\alpha_\tau)) - \epsilon^*(\alpha_\tau) c\right] \mathrm{d}\tau.$$

Notice that it is reasonable to continue funding the project while the expression in square brackets is positive, that is, while

$$\alpha_t \epsilon^*(\alpha_t) R - f(\epsilon^*(\alpha_t)) - \epsilon^*(\alpha_t) c \geqslant 0,$$

or

$$\alpha_t R - c \geqslant \frac{f(\epsilon^*(\alpha_t))}{\epsilon^*(\alpha_t)}.$$

Posterior beliefs can only decrease while experimentation continues, so for some critical $\underline{\alpha}$, $\epsilon^*(\underline{\alpha}) = 0$, and so

$$\underline{\alpha}R - c = \lim_{\alpha\to\underline{\alpha}} \frac{f(\epsilon^*(\alpha))}{\epsilon^*(\alpha)} = \lim_{\alpha\to\underline{\alpha}} \frac{f'(\epsilon^*(\alpha))\,\epsilon^{*\prime}(\alpha)}{\epsilon^{*\prime}(\alpha)} = \lim_{\alpha\to\underline{\alpha}} f'(\epsilon^*(\alpha)) = f'(0) = 0,$$

which immediately gives

$$\underline{\alpha} = \frac{c}{R}$$

—this is the level of optimism at which the funding must stop. Any project with the prior lower than this will not be even considered. Therefore, for any $\alpha \leqslant \underline{\alpha}$,

$$V(\alpha) = 0.$$

Now, for some $\alpha`_t > \alpha_t > \underline{\alpha}$ (where $\alpha`_t$ is just some arbitrary number, not a derivative), for every $\tau \geqslant t$, define

$$\epsilon`_\tau \equiv \frac{\alpha_\tau}{\alpha`_\tau}\epsilon^*(\alpha_\tau).$$

Since initial $\epsilon`_t < \epsilon^*(\alpha_t)$, and since

$$\begin{aligned}\dot{\alpha`}_t &= -\alpha`_t\epsilon`_t(1-\alpha`_t)\\ &= -\alpha`_t\frac{\alpha_t}{\alpha`_t}\epsilon^*(\alpha_t)(1-\alpha`_t)\end{aligned}$$

$$= -\alpha_t \epsilon^* (\alpha_t) (1 - \alpha`_t)$$
$$> -\alpha_t \epsilon^* (\alpha_t) (1 - \alpha_t)$$
$$= \dot{\alpha}_t,$$

then every subsequent $\alpha`_\tau > \alpha_\tau$ for $\tau \geqslant t$, and so every $\epsilon`_\tau < \epsilon^* (\alpha_\tau)$.

This way, if the belief that the project is still good, $\alpha`_t$ is higher than $\alpha_t$, then we can do better just by following funding path $(\epsilon`_t, t \geqslant 0)$:

$$\begin{aligned}
&\mathcal{V} (\alpha`_t) \\
&\geqslant \int_t^\infty e^{-(\tau-t)r - \int_t^\tau \alpha`_\theta \epsilon`_\theta \, \mathrm{d}\theta} \left(\alpha`_\tau \epsilon`_\tau R - f(\epsilon`_\tau) - \epsilon`_\tau c\right) \mathrm{d}\tau \\
&= \int_t^\infty e^{-(\tau-t)r - \int_t^\tau \alpha`_\theta \frac{\alpha_\theta}{\alpha`_\theta} \epsilon^*(\alpha_\theta) \, \mathrm{d}\theta} \left(\alpha`_\tau \frac{\alpha_\tau}{\alpha`_\tau} \epsilon^* (\alpha_\tau) R - f(\epsilon`_\tau) - \epsilon`_\tau c\right) \mathrm{d}\tau \\
&= \int_t^\infty e^{-(\tau-t)r - \int_t^\tau \alpha_\theta \epsilon^*(\alpha_\theta) \, \mathrm{d}\theta} \left(\alpha_\tau \epsilon^* (\alpha_\tau) R - f(\epsilon`_\tau) - \epsilon`_\tau c\right) \mathrm{d}\tau \\
&> \int_t^\infty e^{-(\tau-t)r - \int_t^\tau \alpha_\theta \epsilon^*(\alpha_\theta) \, \mathrm{d}\theta} \left(\alpha_\tau \epsilon^* (\alpha_\tau) R - f(\epsilon^* (\alpha_\tau)) - \epsilon^* (\alpha_\tau) c\right) \mathrm{d}\tau \\
&= \mathcal{V} (\alpha_t)
\end{aligned}$$

because for every $\tau \geqslant t$, $\epsilon`_\tau < \epsilon^* (\alpha_\tau)$, and so

$$-f(\epsilon`_\tau) - \epsilon`_\tau c > -f(\epsilon^* (\alpha_\tau)) - \epsilon^* (\alpha_\tau) c.$$

Therefore, when $\alpha`_t > \alpha_t > \underline{\alpha}$,

$$\mathcal{V} (\alpha`_t) > \mathcal{V} (\alpha_t),$$

and so when $\alpha > \underline{\alpha}$,

$$\mathcal{V}' (\alpha) > 0.$$

The main implication of this is

$$r\mathcal{V}' (\alpha) = f'' (\epsilon^* (\alpha)) \epsilon^* (\alpha) \epsilon^{*\prime} (\alpha) > 0,$$

which is possible only if $\epsilon^{*\prime} (\alpha) > 0$ because for $\alpha > \underline{\alpha}$, $\epsilon^* (\alpha) > 0$ and so $f'' (\epsilon^* (\alpha)) > 0$.

Express $\mathcal{V} (\alpha)$ from (III.5) and $\mathcal{V}' (\alpha)$ from (III.6) and plug these to the first order condition:

$$r (\alpha R - f' (\epsilon^* (\alpha)) - c) - \alpha \left[f'' (\epsilon^* (\alpha)) \epsilon^* (\alpha) \epsilon^{*\prime} (\alpha) (1 - \alpha) + f' (\epsilon^* (\alpha)) \epsilon^* (\alpha) \right.$$

$$- f\left(\epsilon^*\left(\alpha\right)\right)\right] = 0.$$

This is just an ordinary differential equation, which produces an optimal policy function $\epsilon^*\left(\alpha\right)$ when solved given some particular function $f\left(\cdot\right)$.

The boundary condition is simple:

$$\epsilon^*\left(\frac{c}{R}\right) = 0,$$

so this efficient policy function is bounded from below. To see if it's bounded from above, consider $\alpha = 1$, which is the most optimistic belief possible. The ODE becomes

$$r\left(R - c\right) - f'\left(\epsilon^*\left(1\right)\right)\left(r + \epsilon^*\left(1\right)\right) + f\left(\epsilon^*\left(1\right)\right) = 0,$$

or

$$f'\left(\epsilon^*\left(1\right)\right)\left(r + \epsilon^*\left(1\right)\right) - f\left(\epsilon^*\left(1\right)\right) = r\left(R - c\right),$$

which has a unique solution since expression

$$f'\left(\epsilon\right)\left(r + \epsilon\right) - f\left(\epsilon\right)$$

is equal to zero when ϵ is zero, and it strictly increases in ϵ for all other positive values of ϵ because its derivative with respect to ϵ,

$$f''\left(\epsilon\right)\left(r + \epsilon\right) + f'\left(\epsilon\right) - f'\left(\epsilon\right) = f''\left(\epsilon\right)\left(r + \epsilon\right),$$

is strictly positive given strict convexity of f.Therefore function $\epsilon^*\left(\cdot\right)$ is bounded and strictly increasing from zero to $\epsilon^*\left(1\right)$.

It is important to see that all its derivatives are finite. Rearrange terms of the ODE and express

$$\epsilon^{*\prime}\left(\alpha\right) = \frac{r\left(\alpha R - f'\left(\epsilon^*\left(\alpha\right)\right) - c\right) - \alpha\left(f'\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right) - f\left(\epsilon^*\left(\alpha\right)\right)\right)}{\alpha f''\left(\epsilon^*\left(\alpha\right)\right)\epsilon^*\left(\alpha\right)\left(1 - \alpha\right)}.$$

Notice that for $\alpha \in \left(\frac{c}{R}, 1\right)$, the derivative of the policy function is finite, as function $\epsilon^*\left(\alpha\right)$ is bounded, and $f\left(x\right)$ is at least thrice continuously differentiable. However, it is not clear the $\epsilon^{*\prime}\left(\alpha\right)$ is finite on the boundary, when $\alpha = \frac{c}{R}$.

Check:

$$\lim_{\alpha\to\frac{c}{R}} \epsilon^{*\prime}(\alpha) = \lim_{\alpha\to\frac{c}{R}} \frac{r(\alpha R - f'(\epsilon^*(\alpha)) - c) - \alpha(f'(\epsilon^*(\alpha))\epsilon^*(\alpha) - f(\epsilon^*(\alpha)))}{\alpha f''(\epsilon^*(\alpha))\epsilon^*(\alpha)(1-\alpha)}$$
$$= \frac{rR^2\left(R - f''(0)\epsilon^{*\prime}\left(\frac{c}{R}\right)\right)}{f''(0)\epsilon^{*\prime}\left(\frac{c}{R}\right)(R-c)c}.$$

If $\epsilon^{*\prime}\left(\frac{c}{R}\right) = \infty$, then this expression is inconsistent, and so it must be the case that $\epsilon^{*\prime}\left(\frac{c}{R}\right) < \infty$.

III.D. Observable but Unverifiable Effort

This appendix is devoted to the solution of the problem within the "observable but unverifiable effort" framework.

Begin with the entrepreneur's problem. Since $\alpha_t = \hat{\alpha}_t$ for every t, (III.1) becomes

$$rV(\alpha,\alpha) = \max_{\epsilon}\left[\alpha\epsilon sR - f(\epsilon) + \gamma c - \epsilon c - \alpha\epsilon\left(V(\alpha,\hat{\alpha}) + (1-\alpha)\frac{\mathrm{d}V(\alpha,\alpha)}{\mathrm{d}\alpha}\right)\right].$$

The participation constraint is not binding: the entrepreneur can always choose to divert the funds, so $\gamma_t c$ is her guaranteed income. To provide enough incentives for her to experiment, she must expect to do better than that.

The first order condition is

$$\alpha sR - f'(\epsilon^{**}(\alpha)) - c - \alpha\left(V(\alpha,\hat{\alpha}) + (1-\alpha)\frac{\mathrm{d}V(\alpha,\alpha)}{\mathrm{d}\alpha}\right) = 0,$$

where $\epsilon^{**}(\alpha)$ is the policy function for the observable but unverifiable effort environment. The second order condition

$$-f''(\epsilon^{**}(\alpha)) \leqslant 0$$

is satisfied for all $\epsilon \geqslant 0$.

Multiply the both sides of the first order condition by $\epsilon^{**}(\alpha)$ and rearrange:

$$\alpha\epsilon^{**}(\alpha)sR - \epsilon^{**}(\alpha)c - \alpha\epsilon^{**}(\alpha)\left(V(\alpha,\hat{\alpha}) + (1-\alpha)\frac{\mathrm{d}V(\alpha,\alpha)}{\mathrm{d}\alpha}\right) = f'(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha).$$

Combining this with the HJB equation yields

$$rV(\alpha,\alpha) = f'(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) - f(\epsilon^{**}(\alpha)) + \gamma(\alpha)c,$$

and so

$$\begin{aligned} r\frac{\mathrm{d}V\left(\alpha,\alpha\right)}{\mathrm{d}\alpha} &= f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)\epsilon^{**\prime}\left(\alpha\right)+f'\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**\prime}\left(\alpha\right)-f'\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**\prime}\left(\alpha\right)+\gamma'\left(\alpha\right)c \\ &= f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)\epsilon^{**\prime}\left(\alpha\right)+\gamma'\left(\alpha\right)c. \end{aligned}$$

Plug this, together with the expression for the value function, into the first order condition:

$$\begin{gathered} r\left(\alpha s\left(\alpha\right)R-f'\left(\epsilon^{**}\left(\alpha\right)\right)-c\right)-\alpha\left[f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)\epsilon^{**\prime}\left(\alpha\right)\left(1-\alpha\right)\right. \\ \left.+f'\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)-f\left(\epsilon^{**}\left(\alpha\right)\right)+\gamma\left(\alpha\right)c+\gamma'\left(\alpha\right)c\left(1-\alpha\right)\right]=0. \end{gathered}$$

This is the ordinary differential equation that produces a policy function, $\epsilon^{**}\left(\alpha\right)$, given some particular function $f\left(\cdot\right)$.

The investor will want to be on the entrepreneur's optimal path: the entrepreneur does not have the funds of her own, so if at some time t the investor decides to provide less funds, say

$$\gamma_t<\epsilon^{**}\left(\alpha_t\right),$$

then the entrepreneur will have to invest only γ_t. Given the concavity of her maximization problem she will want to invest as close to $\epsilon^{**}\left(\alpha\right)$ as possible. However, this is not the best decision on the investor's side: he could have offered a lower share, s_t, and achieved the same outcome.

Suppose, he suddenly decided to offer more funds:

$$\gamma_t>\epsilon^{**}\left(\alpha\right).$$

This decision will not affect $\epsilon^{**}\left(\alpha\right)$, as γ_t does not affect the instantaneous first order condition, and so the entrepreneur will still invest $\epsilon^{**}\left(\alpha\right)$. The difference

$$\gamma_t-\epsilon^{**}\left(\alpha\right)$$

will be diverted, which is waste from the perspectives of the investor.

Therefore, the investor will always want to satisfy

$$\gamma_t=\epsilon^{**}\left(\alpha\right),$$

and so the incentive constraint implies

$$r\left(\alpha s\left(\alpha\right)R-f'\left(\epsilon^{**}\left(\alpha\right)\right)-c\right)-\alpha\left[f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)\epsilon^{**\prime}\left(\alpha\right)\left(1-\alpha\right)\right.$$
$$\left.+f'\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)-f\left(\epsilon^{**}\left(\alpha\right)\right)+\epsilon^{**}\left(\alpha\right)c+\epsilon^{**\prime}\left(\alpha\right)c\left(1-\alpha\right)\right]=0.$$

Rearrange and express $s\left(\alpha\right)$

$$s\left(\alpha\right)=\frac{\left(f'\left(\epsilon^{**}\left(\alpha\right)\right)+c\right)\left(r+\alpha\epsilon^{**}\left(\alpha\right)\right)-\alpha f\left(\epsilon^{**}\left(\alpha\right)\right)}{\alpha rR}$$
$$+\frac{\alpha\left(1-\alpha\right)\epsilon^{**\prime}\left(\alpha\right)\left(f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)+c\right)}{\alpha rR}.$$

This is everything it is possible to know about the entrepreneur's optimal path at this stage. Now I can solve the investor's problem. Use (III.2) and the entrepreneur's first order condition as an incentive constraint, keeping in mind that $\gamma=\epsilon$ along the desired path:

$$rW\left(\alpha\right)=\max_{s,\,\epsilon}\left[\alpha\epsilon\left(1-s\right)R-\epsilon c-\alpha\epsilon\left(W\left(\alpha\right)+\left(1-\alpha\right)W'\left(\alpha\right)\right)\right]$$

subject to

$$\alpha sR-f'\left(\epsilon\right)-c-\alpha\left(V\left(\alpha,\hat{\alpha}\right)+\left(1-\alpha\right)\frac{\mathrm{d}V\left(\alpha,\alpha\right)}{\mathrm{d}\alpha}\right)=0.$$

Express the problem in the Lagrangian form:

$$rW\left(\alpha\right)=\max_{s,\,\epsilon}\left[\alpha\epsilon\left(1-s\right)R-\epsilon c-\alpha\epsilon\left(W\left(\alpha\right)+\left(1-\alpha\right)W'\left(\alpha\right)\right)\right.$$
$$\left.+\lambda\left(\alpha sR-f'\left(\epsilon\right)-c-\alpha\left(V\left(\alpha,\hat{\alpha}\right)+\left(1-\alpha\right)\frac{\mathrm{d}V\left(\alpha,\alpha\right)}{\mathrm{d}\alpha}\right)\right)\right].$$

There are two first order conditions: with respect to ϵ,

$$\alpha\left(1-s\left(\alpha\right)\right)R-c-\alpha\left(W\left(\alpha\right)+\left(1-\alpha\right)W'\left(\alpha\right)\right)-\lambda f''\left(\epsilon^{**}\left(\alpha\right)\right)=0,$$

and with respect to s,

$$-\alpha\epsilon^{**}\left(\alpha\right)R+\lambda\left(\alpha\right)\alpha R=0.$$

From the second condition, it is obvious that

$$\lambda\left(\alpha\right)=\epsilon^{**}\left(\alpha\right),$$

and so if I use this fact and multiply both sides of the first condition by $\epsilon^{**}(\alpha)$ to produce

$$\alpha\epsilon^{**}(\alpha)(1-s)R - \epsilon^{**}(\alpha)c - \alpha\epsilon^{**}(\alpha)(W(\alpha)+(1-\alpha)W'(\alpha)) = f''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2,$$

I can just plug this expression directly into the HJB equation to conjure

$$rW(\alpha) = f''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2.$$

Therefore,

$$rW'(\alpha) = f'''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2\epsilon^{**\prime}(\alpha) + 2f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha)\epsilon^{**\prime}(\alpha).$$

Combine this, together with the the expression for the value function, with the first order condition number one to produce

$$\begin{aligned}
&r(\alpha(1-s(\alpha))R - f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) - c)\\
&-\alpha\left[f''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2 + (1-\alpha)\left(f'''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2\epsilon^{**\prime}(\alpha)\right.\right.\\
&\left.\left.+2f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha)\epsilon^{**\prime}(\alpha)\right)\right] = 0.
\end{aligned}$$

Plug the expression for $s(\alpha)$ obtained earlier by solving the entrepreneur's problem into this condition to get

$$\begin{aligned}
&r\left(\alpha R - \frac{(f'(\epsilon^{**}(\alpha))+c)(r+\alpha\epsilon^{**}(\alpha)) - \alpha f(\epsilon^{**}(\alpha))}{r}\right.\\
&\left.-\frac{\alpha(1-\alpha)\epsilon^{**\prime}(\alpha)(f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha)+c)}{r} - f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) - c\right)\\
&-\alpha\left[f''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2 + (1-\alpha)\left(f'''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2\epsilon^{**\prime}(\alpha)\right.\right.\\
&\left.\left.+2f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha)\epsilon^{**\prime}(\alpha)\right)\right] = 0,
\end{aligned}$$

or

$$\begin{aligned}
&r(\alpha R - c) - [f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) + f'(\epsilon^{**}(\alpha)) + c](r+\alpha\epsilon^{**}(\alpha)) + \alpha f(\epsilon^{**}(\alpha))\\
&= \alpha(1-\alpha)\epsilon^{**\prime}(\alpha)\left[f'''(\epsilon^{**}(\alpha))(\epsilon^{**}(\alpha))^2 + 3f''(\epsilon^{**}(\alpha))\epsilon^{**}(\alpha) + c\right],
\end{aligned}$$

which is an ODE that produces $\epsilon^{**}(\alpha)$ when solved given some particular function $f(\cdot)$.

For the unique solution to this ODE to exist, I need to define the boundary condition. Plug $\epsilon^{**}(\alpha)=0$ to the ODE to find the desired lowest possible $\underline{\alpha}$, which still makes it possible to agree on continuing with the project:

$$r\left(\underline{\alpha}R-2c\right)=\underline{\alpha}\left(1-\underline{\alpha}\right)\epsilon^{**\prime}\left(\underline{\alpha}\right)c.$$

The left-hand side of this expression is positive only if $\underline{\alpha}\geqslant\frac{2c}{R}$, the right-hand side is positive always: $\epsilon^{**\prime}\left(\underline{\alpha}\right)<0$ would be impossible since $\epsilon^{**}\left(\underline{\alpha}\right)=0$. Therefore, there is always a room for agreement as long as $\alpha\geqslant\frac{2c}{R}$, which implies $\underline{\alpha}=\frac{2c}{R}$ and so the boundary condition is

$$\epsilon^{**}\left(\frac{2c}{R}\right)=0.$$

To get the upper bound, consider $\alpha=1$:

$$r\left(R-c\right)=\left[f''\left(\epsilon^{**}\left(1\right)\right)\epsilon^{**}\left(1\right)+f'\left(\epsilon^{**}\left(1\right)\right)+c\right]\left(r+\epsilon^{**}\left(1\right)\right)-f\left(\epsilon^{**}\left(1\right)\right),\tag{III.c}$$

which has a unique solution.

Policy function $\epsilon^{**}(\alpha)$ is strictly increasing on $\left(\frac{2c}{R},1\right)$. To see why, suppose there are extreme points of function ϵ^{**} somewhere on $\left(\frac{2c}{R},1\right)$. If they actually exist it can only mean that the function must decrease somewhere. Differentiate the ODE with respect to α and assume $\epsilon^{**\prime}(\alpha)=0$:

$$\begin{aligned}&rR-\left[f''\left(\epsilon^{**}(\alpha)\right)\epsilon^{**}(\alpha)+f'\left(\epsilon^{**}(\alpha)\right)+c\right]\left(r+\epsilon^{**}(\alpha)\right)+f\left(\epsilon^{**}(\alpha)\right)\\&=\alpha\left(1-\alpha\right)\epsilon^{**\prime\prime}(\alpha)\left[f'''\left(\epsilon^{**}(\alpha)\right)\left(\epsilon^{**}(\alpha)\right)^2+3f''\left(\epsilon^{**}(\alpha)\right)\epsilon^{**}(\alpha)+c\right],\end{aligned}$$

use the same assumption of $\epsilon^{**\prime}(\alpha)=0$ for the ODE itself:

$$r\left(\alpha R-c\right)-\left[f''\left(\epsilon^{**}(\alpha)\right)\epsilon^{**}(\alpha)+f'\left(\epsilon^{**}(\alpha)\right)+c\right]\left(r+\alpha\epsilon^{**}(\alpha)\right)+\alpha f\left(\epsilon^{**}(\alpha)\right)=0,$$

or

$$\begin{aligned}&rR-\left[f''\left(\epsilon^{**}(\alpha)\right)\epsilon^{**}(\alpha)+f'\left(\epsilon^{**}(\alpha)\right)+c\right]\epsilon^{**}(\alpha)+f\left(\epsilon^{**}(\alpha)\right)\\&=r\frac{c}{\alpha}+\left[f''\left(\epsilon^{**}(\alpha)\right)\epsilon^{**}(\alpha)+f'\left(\epsilon^{**}(\alpha)\right)+c\right]\frac{r}{\alpha},\end{aligned}$$

and plug it into the derivative of the ODE:

$$\frac{r}{\alpha}\left[(1-\alpha)\left(f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)+f'\left(\epsilon^{**}\left(\alpha\right)\right)+c\right)+c\right]$$
$$=\alpha\left(1-\alpha\right)\epsilon^{**\prime\prime}\left(\alpha\right)\left[f'''\left(\epsilon^{**}\left(\alpha\right)\right)\left(\epsilon^{**}\left(\alpha\right)\right)^{2}+3f''\left(\epsilon^{**}\left(\alpha\right)\right)\epsilon^{**}\left(\alpha\right)+c\right].$$

The left-hand side of the expression is strictly positive, so the right-hand side must be too, however, it can only be possible if $\epsilon^{**\prime\prime}\left(\alpha\right)>0$, which implies that all the extreme points that can be found for function $\epsilon^{**}\left(\alpha\right)$ are local minima. However, this is impossible: either there is at least one local minimum without local maxima, but then it means that the function can have negative values as it starts from $\epsilon^{**}\left(\frac{2c}{R}\right)=0$; or there must be at least one local maximum due to the smoothness of the function, which is at odds with having local minima only.

Therefore, function $\epsilon^{**}\left(\alpha\right)$ strictly increases everywhere on the interior of its range.

III.E. Unobservable Effort Environment

This appendix is devoted to the characterization of the equilibrium in the unobservable effort environment.

Begin by considering the entrepreneur's problem expressed in the for of HJB equation:

$$rV\left(\alpha,\hat{\alpha}\right)=\max_{\epsilon}\left[\left(\alpha\epsilon sR-f\left(\epsilon\right)-\epsilon c+\gamma c\right)-\alpha\epsilon\left(V_{1}\left(\alpha,\hat{\alpha}\right)\left(1-\alpha\right)+V\left(\alpha,\hat{\alpha}\right)\right)\right.$$
$$\left.-V_{2}\left(\alpha,\hat{\alpha}\right)\hat{\alpha}\hat{\epsilon}\left(\hat{\alpha}\right)\left(1-\hat{\alpha}\right)\right].$$

The corresponding first order condition is

$$\alpha sR-f'\left(\epsilon^{**}\right)-c-\alpha\left[V_{1}\left(\alpha,\hat{\alpha}\right)\left(1-\alpha\right)+V\left(\alpha,\hat{\alpha}\right)\right]=0.$$

The second order condition is satisfied automatically.

The envelope condition is

$$rV_{1}\left(\alpha,\hat{\alpha}\right)=\epsilon^{**}\left(\alpha,\hat{\alpha}\right)s\left(\hat{\alpha}\right)R-\epsilon^{**}\left(\alpha,\hat{\alpha}\right)\left[V_{1}\left(\alpha,\hat{\alpha}\right)\left(1-\alpha\right)+V\left(\alpha,\hat{\alpha}\right)\right]$$
$$-\alpha\epsilon^{**}\left(\alpha,\hat{\alpha}\right)V_{11}\left(\alpha,\hat{\alpha}\right)\left(1-\alpha\right)-V_{12}\left(\alpha,\hat{\alpha}\right)\hat{\alpha}\hat{\epsilon}\left(\hat{\alpha}\right)\left(1-\hat{\alpha}\right).$$

Differentiate both sides of the first order condition with respect to α:

$$s(\hat{\alpha})R - f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon_1^{**}(\alpha,\hat{\alpha}) - [V_1(\alpha,\hat{\alpha})(1-\alpha) + V(\alpha,\hat{\alpha})]$$
$$-\alpha V_{11}(\alpha,\hat{\alpha})(1-\alpha) = 0$$

and plug the result into the envelope condition:

$$rV_1(\alpha,\hat{\alpha}) = f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon^{**}(\alpha,\hat{\alpha})\epsilon_1^{**}(\alpha,\hat{\alpha}) - V_{12}(\alpha,\hat{\alpha})\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha}).$$

Now, differentiate both sides of the first order condition with respect to $\hat{\alpha}$:

$$\alpha s'(\hat{\alpha})R - f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon_2^{**}(\alpha,\hat{\alpha}) - \alpha[V_{12}(\alpha,\hat{\alpha})(1-\alpha) + V_2(\alpha,\hat{\alpha})] = 0,$$

express $V_{12}(\alpha,\hat{\alpha})$

$$V_{12}(\alpha,\hat{\alpha}) = \frac{\alpha s'(\hat{\alpha})R - f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon_2^{**}(\alpha,\hat{\alpha}) - \alpha V_2(\alpha,\hat{\alpha})}{\alpha(1-\alpha)},$$

and insert it into the envelope condition as well:

$$rV_1(\alpha,\hat{\alpha}) = f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon^{**}(\alpha,\hat{\alpha})\epsilon_1^{**}(\alpha,\hat{\alpha})$$
$$-\frac{\alpha s'(\hat{\alpha})R - f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon_2^{**}(\alpha,\hat{\alpha}) - \alpha V_2(\alpha,\hat{\alpha})}{\alpha(1-\alpha)}\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha}).$$

Rearrange:

$$-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})V_2(\alpha,\hat{\alpha}) = (1-\alpha)f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon^{**}(\alpha,\hat{\alpha})\epsilon_1^{**}(\alpha,\hat{\alpha})$$
$$-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})s'(\hat{\alpha})R + \frac{\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon_2^{**}(\alpha,\hat{\alpha})}{\alpha} - (1-\alpha)rV_1(\alpha,\hat{\alpha}),$$

and insert the result into the HJB equation together with the first order condition:

$$r[V(\alpha,\hat{\alpha}) + (1-\alpha)V_1(\alpha,\hat{\alpha})] = f'(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon^{**}(\alpha,\hat{\alpha}) - f(\epsilon^{**}(\alpha,\hat{\alpha})) + \gamma(\hat{\alpha})c$$
$$+(1-\alpha)f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon^{**}(\alpha,\hat{\alpha})\epsilon_1^{**}(\alpha,\hat{\alpha}) - \hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})s'(\hat{\alpha})R$$
$$+\frac{\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})f''(\epsilon^{**}(\alpha,\hat{\alpha}))\epsilon_2^{**}(\alpha,\hat{\alpha})}{\alpha}.$$

From the first order condition, it is clear that

$$V(\alpha,\hat{\alpha}) + V_1(\alpha,\hat{\alpha})(1-\alpha) = s(\hat{\alpha})R - \frac{f'(\epsilon^{**}(\alpha,\hat{\alpha})) + c}{\alpha}.$$

Therefore, the equilibrium condition for the entrepreneur is

$$r\left[s(\hat{\alpha})R-\frac{f'(\epsilon^{**}(\alpha,\hat{\alpha}))+c}{\alpha}\right]=f'(\epsilon^{**}(\alpha,\hat{\alpha}))\,\epsilon^{**}(\alpha,\hat{\alpha})-f(\epsilon^{**}(\alpha,\hat{\alpha}))+\gamma(\hat{\alpha})\,c$$
$$+(1-\alpha)\,f''(\epsilon^{**}(\alpha,\hat{\alpha}))\,\epsilon^{**}(\alpha,\hat{\alpha})\,\epsilon_1^{**}(\alpha,\hat{\alpha})-\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})\,s'(\hat{\alpha})\,R$$
$$+\frac{\hat{\alpha}\hat{\epsilon}(\hat{\alpha})(1-\hat{\alpha})\,f''(\epsilon^{**}(\alpha,\hat{\alpha}))\,\epsilon_2^{**}(\alpha,\hat{\alpha})}{\alpha}.$$

The principal will always provide the agent with the sum that he expects her to allocate towards experiments. The argumentation for this is the same as provided for the observable but unverifiable case.

Hence, from the principals perspective, the agent will always want to be on the path characterized by

$$r\left[s(\hat{\alpha})R-\frac{f'(\hat{\epsilon}^{**}(\hat{\alpha}))+c}{\hat{\alpha}}\right]=f'(\hat{\epsilon}^{**}(\hat{\alpha}))\,\hat{\epsilon}^{**}(\hat{\alpha})-f(\hat{\epsilon}^{**}(\hat{\alpha}))+\hat{\epsilon}^{**}(\hat{\alpha})\,c$$
$$+(1-\hat{\alpha})\,f''(\hat{\epsilon}^{**}(\hat{\alpha}))\,\hat{\epsilon}^{**}(\hat{\alpha})\,\hat{\epsilon}^{**\prime}(\hat{\alpha})-\hat{\alpha}\hat{\epsilon}^{**}(\hat{\alpha})(1-\hat{\alpha})\,s'(\hat{\alpha})\,R. \tag{III.d}$$

Here, I implicitly used the fact that

$$\hat{\epsilon}^{**\prime}(\hat{\alpha})\equiv\frac{\mathrm{d}\hat{\epsilon}^{**}(\hat{\alpha})}{\mathrm{d}\hat{\alpha}}\equiv\frac{\mathrm{d}\epsilon^{**}(\hat{\alpha},\hat{\alpha})}{\mathrm{d}\hat{\alpha}}\equiv\epsilon_1^{**}(\hat{\alpha},\hat{\alpha})+\epsilon_2^{**}(\hat{\alpha},\hat{\alpha}).$$

Rewrite the IC constraint in the form:

$$-\hat{\alpha}\hat{\epsilon}^{**}(\hat{\alpha})(1-\hat{\alpha})\,s'(\hat{\alpha})=rs(\hat{\alpha})-r\frac{f'(\hat{\epsilon}^{**}(\hat{\alpha}))+c}{\hat{\alpha}R}-\frac{1}{R}f'(\hat{\epsilon}^{**}(\hat{\alpha}))\,\hat{\epsilon}^{**}(\hat{\alpha})$$
$$+\frac{1}{R}f(\hat{\epsilon}^{**}(\hat{\alpha}))-\frac{1}{R}\hat{\epsilon}^{**}(\hat{\alpha})\,c-\frac{1}{R}(1-\hat{\alpha})\,f''(\hat{\epsilon}^{**}(\hat{\alpha}))\,\hat{\epsilon}^{**}(\hat{\alpha})\,\hat{\epsilon}^{**\prime}(\hat{\alpha}).$$

Notice that

$$-\hat{\alpha}_t\hat{\epsilon}^{**}(\hat{\alpha}_t)(1-\hat{\alpha}_t)\,s'(\hat{\alpha}_t)=\dot{s}_t,$$

and define

$$h_t\equiv-r\frac{f'(\hat{\epsilon}^{**}(\hat{\alpha}_t))+c}{\hat{\alpha}_t}-f'(\hat{\epsilon}^{**}(\hat{\alpha}_t))\,\hat{\epsilon}^{**}(\hat{\alpha}_t)+f(\hat{\epsilon}^{**}(\hat{\alpha}_t))-\hat{\epsilon}^{**}(\hat{\alpha}_t)\,c,$$

$$g_t\equiv-(1-\hat{\alpha}_t)\,f''(\hat{\epsilon}^{**}(\hat{\alpha}_t))\,\hat{\epsilon}^{**}(\hat{\alpha}_t)\,\hat{\epsilon}^{**\prime}(\hat{\alpha}_t).$$

Then

$$\dot{s}_t = rs_t + \frac{h_t + g_t}{R},$$

which is a linear first order ordinary differential equation with the solution of the form

$$s_t = me^{rt} + e^{rt}\frac{1}{R}\int_0^t e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau,$$

where m is some constant. Then for some other time $T > t$,

$$\begin{aligned}
s_T &= me^{rT} + e^{rT}\frac{1}{R}\int_0^T e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau \\
&= me^{rt}e^{r(T-t)} + e^{rt}e^{r(T-t)}\frac{1}{R}\int_0^T e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau \\
&= e^{r(T-t)}\left[me^{rt} + e^{rt}\frac{1}{R}\int_0^t e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau + e^{rt}\frac{1}{R}\int_t^T e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau\right] \\
&= e^{r(T-t)}s_t + e^{rT}\frac{1}{R}\int_t^T e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau.
\end{aligned}$$

This expression is backward-looking. To make it forward-looking, rearrange:

$$s_t = \frac{s_T}{e^{r(T-t)}} - e^{rt}\frac{1}{R}\int_t^T e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau.$$

Function $s(\cdot)$ is bounded, its range belongs to $[0, 1]$, so it is safe to take limits:

$$\begin{aligned}
s_t &= \lim_{T\to\infty}\left[\frac{s_T}{e^{r(T-t)}} - e^{rt}\frac{1}{R}\int_t^T e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau\right] \\
&= -e^{rt}\frac{1}{R}\int_t^\infty e^{-r\tau}\left(h_\tau + g_\tau\right)\mathrm{d}\tau.
\end{aligned}$$

Let us take a closer look at the second term inside the integral:

$$\begin{aligned}
\int_t^\infty e^{-r\tau}g_\tau\,\mathrm{d}\tau &= -\int_t^\infty e^{-r\tau}\left(1-\hat{\alpha}_\tau\right)f''\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\hat{\epsilon}^{**\prime}\left(\hat{\alpha}_\tau\right)\mathrm{d}\tau \\
&= -\int_t^\infty \frac{e^{-r\tau}}{\hat{\alpha_\tau}}\hat{\alpha}_\tau\left(1-\hat{\alpha}_\tau\right)f''\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\hat{\epsilon}^{**\prime}\left(\hat{\alpha}_\tau\right)\mathrm{d}\tau \\
&= \int_t^\infty \frac{e^{-r\tau}}{\hat{\alpha_\tau}}\frac{\mathrm{d}f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)}{\mathrm{d}\tau}\mathrm{d}\tau \\
&= \frac{e^{-r\tau}}{\hat{\alpha_\tau}}f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)\Bigg|_t^\infty \\
&\quad - \int_t^\infty\left(-r\frac{e^{-r\tau}}{\hat{\alpha_\tau}} + \frac{e^{-r\tau}}{\hat{\alpha_\tau}^2}\hat{\alpha_\tau}\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\left(1-\hat{\alpha}_\tau\right)\right)f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)\mathrm{d}\tau
\end{aligned}$$

$$= -e^{-rt}\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t} - \int_t^\infty e^{-r\tau}\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\left(1-\hat{\alpha}_\tau\right)-r\right)\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)}{\hat{\alpha}_\tau}\,\mathrm{d}\tau.$$

Therefore,

$$\begin{aligned}
&\int_t^\infty e^{-r\tau}\left(h_\tau+g_\tau\right)\mathrm{d}\tau = \int_t^\infty e^{-r\tau}h_\tau\,\mathrm{d}\tau + \int_t^\infty e^{-r\tau}g_\tau\,\mathrm{d}\tau \\
&= \int_t^\infty e^{-r\tau}\left(-r\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)+c}{\hat{\alpha}_\tau} - f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right) + f\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right) - \hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)c\right)\mathrm{d}\tau \\
&\quad - e^{-rt}\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t} - \int_t^\infty e^{-r\tau}\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\left(1-\hat{\alpha}_\tau\right)-r\right)\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)}{\hat{\alpha}_\tau}\,\mathrm{d}\tau \\
&= \int_t^\infty e^{-r\tau}\left(f\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right) - \hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\left(\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)}{\hat{\alpha}_\tau}+c\right) - r\frac{c}{\hat{\alpha}_\tau}\right)\mathrm{d}\tau - e^{-rt}\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t}.
\end{aligned}$$

And so,

$$\begin{aligned}
s_t &= -e^{rt}\frac{1}{R}\int_t^\infty e^{-r\tau}\left(h_\tau+g_\tau\right)\mathrm{d}\tau \\
&= -e^{rt}\frac{1}{R}\left[\int_t^\infty e^{-r\tau}\left(f\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right) - \hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\left(\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)}{\hat{\alpha}_\tau}+c\right) - r\frac{c}{\hat{\alpha}_\tau}\right)\mathrm{d}\tau \right.\\
&\left. - e^{-rt}\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t}\right] \\
&= \frac{1}{R}\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t} - \frac{1}{R}\int_t^\infty e^{-(\tau-t)r}\left(f\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right) - \hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\left(\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_\tau\right)\right)}{\hat{\alpha}_\tau}+c\right) - r\frac{c}{\hat{\alpha}_\tau}\right)\mathrm{d}\tau.
\end{aligned}$$

Define

$$k_t \equiv f\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right) - \hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\left(\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t}+c\right) - r\frac{c}{\hat{\alpha}_t},$$

so that

$$s_t = \frac{1}{R}\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t} - \frac{1}{R}\int_t^\infty e^{-(\tau-t)r}k_\tau\,\mathrm{d}\tau.$$

This expression describes the share that satisfies the incentive constrain conditional on the funding path.

Now that the investor knows what the entrepreneur's optimal path is expected to be based on the investor's offer schedule, he can solve his problem having the entrepreneur's first order condition as a constraint. Use (III.2) to state the problem:

$$rW\left(\hat{\alpha}\right) = \max_{s,\,\hat{\epsilon}}\left[\hat{\alpha}\hat{\epsilon}\left(1-s\right)R - \hat{\epsilon}c - \hat{\alpha}\hat{\epsilon}\left(W\left(\hat{\alpha}\right)+\left(1-\hat{\alpha}\right)W'\left(\hat{\alpha}\right)\right)\right]$$

subject to

$$\hat{\alpha} s R - f'(\hat{\epsilon}) - c - \alpha \left[V_1(\hat{\alpha}, \hat{\alpha})(1-\hat{\alpha}) + V(\hat{\alpha}, \hat{\alpha}) \right] = 0,$$

and the participation constraint that I will ignore for now, but verify that it is satisfied once the optimality conditions are set.

In the Lagrangian form, the problem becomes

$$\begin{aligned} rW(\hat{\alpha}) = \max_{s, \hat{\epsilon}} & \left[\hat{\alpha}\hat{\epsilon}(1-s) R - \hat{\epsilon} c - \hat{\alpha}\hat{\epsilon}\left(W(\hat{\alpha}) + (1-\hat{\alpha}) W'(\hat{\alpha}) \right) \right. \\ & \left. + \lambda \left(\hat{\alpha} s R - f'(\hat{\epsilon}) - c - \alpha \left[V_1(\hat{\alpha}, \hat{\alpha})(1-\hat{\alpha}) + V(\hat{\alpha}, \hat{\alpha}) \right] \right) \right] \end{aligned}$$

There are two first order conditions. The first one, with respect to $\hat{\epsilon}$:

$$\hat{\alpha}(1 - s(\hat{\alpha})) R - c - \hat{\alpha}\left(W(\hat{\alpha}) + (1-\hat{\alpha}) W'(\hat{\alpha}) \right) - \lambda f''(\hat{\epsilon}^{**}(\hat{\alpha})) = 0,$$

and the second one, with respect to s:

$$-\hat{\alpha}\hat{\epsilon}^{**}(\hat{\alpha}) R + \lambda(\hat{\alpha}) \hat{\alpha} R = 0.$$

From the second condition, I immediately get

$$\lambda(\hat{\alpha}) = \hat{\epsilon}^{**}(\hat{\alpha}).$$

Use this result in the first condition, rearrange the terms, and multiply both of its sides by $\hat{\epsilon}^{**}(\hat{\alpha})$:

$$\begin{aligned} \hat{\alpha}\hat{\epsilon}^{**}(\hat{\alpha})(1 - s(\hat{\alpha})) R - \hat{\epsilon}^{**}(\hat{\alpha}) c - \hat{\alpha}\hat{\epsilon}^{**}(\hat{\alpha})\left(W(\hat{\alpha}) + (1-\hat{\alpha}) W'(\hat{\alpha}) \right) \\ = f''(\hat{\epsilon}^{**}(\hat{\alpha}))\left(\hat{\epsilon}^{**}(\hat{\alpha})\right)^2. \end{aligned}$$

Hence

$$rW(\hat{\alpha}) = f''(\hat{\epsilon}^{**}(\hat{\alpha}))\left(\hat{\epsilon}^{**}(\hat{\alpha})\right)^2,$$

which is positive for positive funding rates, hence no need for the explicit participation constraint, and

$$rW'(\hat{\alpha}) = f'''(\hat{\epsilon}^{**}(\hat{\alpha}))\left(\hat{\epsilon}^{**}(\hat{\alpha})\right)^2 \hat{\epsilon}^{**\prime}(\hat{\alpha}) + 2 f''(\hat{\epsilon}^{**}(\hat{\alpha}))\, \hat{\epsilon}^{**}(\hat{\alpha})\, \hat{\epsilon}^{**\prime}(\hat{\alpha}).$$

Plugging these results back into the first first order condition, I obtain

$$r\left(\hat{\alpha}\left(1-s\left(\hat{\alpha}\right)\right)R-f''\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)\hat{\epsilon}^{**}\left(\hat{\alpha}\right)-c\right)$$
$$-\hat{\alpha}\left[f''\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)^{2}+\left(1-\hat{\alpha}\right)\left(f'''\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)^{2}\hat{\epsilon}^{**\prime}\left(\hat{\alpha}\right)\right.\right.$$
$$\left.\left.+\,2f''\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\hat{\epsilon}^{**\prime}\left(\hat{\alpha}\right)\right)\right]=0. \tag{III.e}$$

Funding rate is bounded below if it eventually decreases:

$$\lim_{t\to\infty}\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)=0.$$

Consider the limit of the share function as time advances:

$$\begin{aligned}\lim_{t\to\infty}s_t&=\lim_{t\to\infty}\left[\frac{1}{R}\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t}-\frac{1}{R}\int_t^{\infty}e^{-(\tau-t)r}k_\tau\,\mathrm{d}\tau\right]\\&=-\frac{1}{R}\lim_{t\to\infty}\frac{\int_t^{\infty}e^{-\tau r}k_\tau\,\mathrm{d}\tau}{e^{-tr}}\\&=-\frac{1}{R}\lim_{t\to\infty}\frac{e^{-tr}k_t}{e^{-tr}r}\\&=-\frac{1}{R}\lim_{t\to\infty}\frac{f\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)-\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\left(\frac{f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}_t\right)\right)}{\hat{\alpha}_t}+c\right)-r\frac{c}{\hat{\alpha}_t}}{r}\\&=\frac{c}{\underline{\hat{\alpha}}R}.\end{aligned}$$

So, $s\left(\underline{\hat{\alpha}}\right)=\frac{c}{\underline{\hat{\alpha}}R}$. Use this in differential equation (III.e) while also taking limits:

$$r\left(\underline{\hat{\alpha}}\left(1-\frac{c}{\underline{\hat{\alpha}}R}\right)R-c\right)=0,$$

or simply

$$\underline{\hat{\alpha}}=\frac{2c}{R}.$$

This is the lower bound.

To combine the two conditions into one, first, differentiate III.e with respect to $\hat{\alpha}$ and express $s'\left(\hat{\alpha}\right)$; second, plug this result together with III.e into III.d to produce the second order differential equation that can be solved for the optimal unobserved funding rate, given some particular function $f\left(\cdot\right)$:

$$r\left(\hat{\alpha}R-f'\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)-f''\left(\hat{\epsilon}^{**}\left(\hat{\alpha}\right)\right)\hat{\epsilon}^{**}\left(\hat{\alpha}\right)-2c\right)$$

$$
\begin{aligned}
&- \hat{\alpha} \left[f' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \hat{\epsilon}^{**} \left(\hat{\alpha} \right) - f \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) + f'' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right)^2 + \hat{\epsilon}^{**} \left(\hat{\alpha} \right) c \right] \\
&+ \left(1 - \hat{\alpha} \right) \left[f'' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right)^2 + \hat{\epsilon}^{**} \left(\hat{\alpha} \right) c \right] \\
&- \hat{\alpha} \left(1 - \hat{\alpha} \right) \left[f''' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right)^2 + 2 f'' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right] 2 \hat{\epsilon}^{**\prime} \left(\hat{\alpha} \right) \\
&- \frac{\hat{\alpha}^2 \hat{\epsilon}^{**} \left(\hat{\alpha} \right) \left(1 - \hat{\alpha} \right)^2}{r} \left[f''' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right)^2 + 2 f'' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right] \hat{\epsilon}^{**\prime\prime} \left(\hat{\alpha} \right) \\
&- \frac{\hat{\alpha}^2 \hat{\epsilon}^{**} \left(\hat{\alpha} \right) \left(1 - \hat{\alpha} \right)^2}{r} \left[f'''' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right)^2 + 4 f''' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \hat{\epsilon}^{**} \left(\hat{\alpha} \right) + 2 f'' \left(\hat{\epsilon}^{**} \left(\hat{\alpha} \right) \right) \right] \left(\hat{\epsilon}^{**\prime} \left(\hat{\alpha} \right) \right)^2 \\
&= 0.
\end{aligned}
$$

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